

Polynomial Functions

6A Operations with Polynomials

- 6-1 Polynomials
- 6-2 Multiplying Polynomials
- 6-3 Dividing Polynomials
- Lab Explore the Sum and Difference of Two Cubes
- 6-4 Factoring Polynomials

MULTI-STEP TEST PREP

6B Applying Polynomial Functions

- 6-5 Finding Real Roots of Polynomial Equations
- 6-6 Fundamental Theorem of Algebra
- Lab Explore End Behavior
- 6-7 Investigating Graphs of Polynomial Functions
- 6-8 Transforming Polynomial Functions
- 6-9 Curve Fitting with Polynomial Models

MULTI-STEP TEST PREP

FILL IT UP!

You can use polynomials to predict the shape of containers.



Chapter Project Online

KEYWORD: MB7 ChProj

ARE YOU READY?

Vocabulary

Match each term on the left with a definition on the right.

- | | |
|--------------------------|--|
| 1. coefficient | A. the y-value of the highest point on the graph of the function |
| 2. like terms | B. the horizontal number line that divides the coordinate plane |
| 3. root of an equation | C. the numerical factor in a term |
| 4. x-intercept | D. a value of the variable that makes the equation true |
| 5. maximum of a function | E. terms that contain the same variables raised to the same powers |
| | F. the x-coordinate of a point where a graph intersects the x-axis |

Evaluate Powers

Evaluate each expression.

6. 6^4 7. -5^4 8. $(-1)^5$ 9. $\left(-\frac{2}{3}\right)^2$

Evaluate Expressions

Evaluate each expression for the given value of the variable.

10. $x^4 - 5x^2 - 6x - 8$ for $x = 3$ 11. $2x^3 - 3x^2 - 29x - 30$ for $x = -2$
12. $2x^3 - x^2 - 8x + 4$ for $x = \frac{1}{2}$ 13. $3x^4 + 5x^3 + 6x^2 + 4x - 1$ for $x = -1$

Multiply and Divide Monomials

Multiply or divide.

14. $2x^3y \cdot 4x^2$ 15. $-5a^2b \cdot ab^4$ 16. $\frac{-7t^4}{3t^2}$ 17. $\frac{3p^3q^2r}{12pr^4}$

Surface Area

Find the surface area of each solid.

18. cube with side length 4 cm
19. rectangular prism with height 3 ft, width 1.5 ft, and length 8 ft

Volume

Find the volume of each solid.

20. rectangular prism with height 1 in., width 6 in., and length $\frac{2}{3}$ in.
21. rectangular prism with height 5 cm and a square base with side length 2 cm

Study Guide: Preview

Where You've Been

Previously, you

- used transformations to graph quadratic functions.
- solved quadratic equations.
- used the Zero Product Property to find the zeros of quadratic functions.
- modeled data with quadratic models.

In This Chapter

You will study

- using transformations to graph polynomial functions.
- solving polynomial equations.
- the zeros of polynomial functions.
- modeling data with polynomial models.

Where You're Going

You can use the skills in this chapter

- to solve problems in future math classes, including College Algebra and Trigonometry.
- to solve real-life problems in physics and graphic arts.
- to predict the value of stocks.
- to maximize or minimize volume and area.

Key Vocabulary/Vocabulario

end behavior	comportamiento extremo
leading coefficient	coeficiente principal
local maximum	máximo local
local minimum	mínimo local
monomial	monomio
multiplicity	multiplicidad
polynomial	polinomio
polynomial function	función polinomial
synthetic division	división sintética
turning point	punto de inflexión

Vocabulary Connections

To become familiar with some of the vocabulary terms in the chapter, consider the following. You may refer to the chapter, the glossary, or a dictionary if you like.

1. In what position would you find the *leading* runner in a race? In what position do you suppose you would find the **leading coefficient** in a polynomial?
2. A **local minimum** of a function is a value less than any other value in the region around it. Which of the vocabulary terms do you think describes the value of a function that is greater than any other value in the region around it?
3. The word **monomial** begins with the root *mono-*. List some other words that begin with *mono-*. What do all of these words have in common?
4. The everyday meaning of **turning point** is “a point at which a change takes place.” What might happen at a *turning point* on the graph of a polynomial function?

Study Strategy: Remember Theorems and Formulas

In math, there are many formulas, properties, theorems, and rules that you must commit to memory. To help you remember an important rule, write it on an index card. Include a diagram or an example, and add notes about the important details. Study your index cards on a regular basis.

From Lesson 5-6

The Quadratic Formula

If $ax^2 + bx + c = 0$ ($a \neq 0$), then the solutions, or roots, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sample Index Card

Quadratic Formula

If $ax^2 + bx + c = 0$ ($a \neq 0$), then the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- This can be used to solve any quadratic equation.
- Before using the formula, make sure the equation is written in standard form.

$$\bullet f(x) = x^2 + 2x - 24 \rightarrow x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-24)}}{2(1)}$$

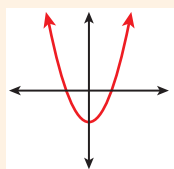
Try This

1. Create index cards for the discriminant formulas shown in the table below.
2. Explain why you need to understand the principles and concepts of the quadratic formula prior to memorizing the discriminant properties.
3. Describe a plan to help you memorize the quadratic formula and the discriminant formulas.

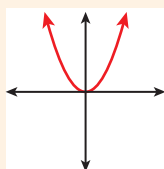
Discriminant

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) is $b^2 - 4ac$.

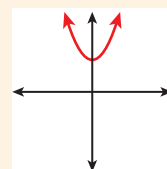
If $b^2 - 4ac > 0$, the equation has two distinct real solutions.



If $b^2 - 4ac = 0$, the equation has one distinct real solution.



If $b^2 - 4ac < 0$, the equation has two distinct nonreal complex solutions.





6-1

Polynomials

Objectives

Identify, evaluate, add, and subtract polynomials.
Classify and graph polynomials.

Vocabulary

monomial
polynomial
degree of a monomial
degree of a polynomial
leading coefficient
binomial
trinomial
polynomial function

Who uses this?

Doctors can use polynomials to model blood flow. (See Example 4.)



A **monomial** is a number or a product of numbers and variables with whole number exponents. A **polynomial** is a monomial or a sum or difference of monomials. Each monomial in a polynomial is a term. Because a monomial has only one term, it is the simplest type of polynomial.

Polynomials have no variables in denominators or exponents, no roots or absolute values of variables, and all variables have whole number exponents.

Polynomials: $3x^4$ $2z^{12} + 9z^3$ $\frac{1}{2}a^7$ $0.15x^{101}$ $3t^2 - t^3$

Not polynomials: 3^x $|2b^3 - 6b|$ $\frac{8}{5y^2}$ $\frac{1}{2}\sqrt{x}$ $m^{0.75} - m$

The **degree of a monomial** is the sum of the exponents of the variables.

EXAMPLE 1 Identifying the Degree of a Monomial

Identify the degree of each monomial.

A x^4
 x^4 *Identify the exponent.*
The degree is 4.

B 12
 $12 = 12x^0$ *Identify the exponent.*
The degree is 0.

C $4a^2b$
 $4a^2b^1$ *Add the exponents.*
The degree is 3.

D x^3y^4z
 $x^3y^4z^1$ *Add the exponents.*
The degree is 8.



Identify the degree of each monomial.

1a. x^3

1b. 7

1c. $5x^3y^2$

1d. a^6bc^2

The **degree of a polynomial** is given by the term with the greatest degree. A polynomial with one variable is in standard form when its terms are written in descending order by degree. So, in standard form, the degree of the first term indicates the degree of the polynomial, and the **leading coefficient** is the coefficient of the first term.

Standard Form

Leading coefficient Degree of polynomial

$5x^3 + 8x^2 + 3x - 17$

Degree of term: 3 2 1 0

A polynomial can be classified by its number of terms. A polynomial with two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**. A polynomial can also be classified by its degree.



Classifying Polynomials by Degree		
Name	Degree	Example
Constant	0	-9
Linear	1	$x - 4$
Quadratic	2	$x^2 + 3x - 1$
Cubic	3	$x^3 + 2x^2 + x + 1$
Quartic	4	$2x^4 + x^3 + 3x^2 + 4x - 1$
Quintic	5	$7x^5 + x^4 - x^3 + 3x^2 + 2x - 1$

EXAMPLE 2 Classifying Polynomials

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

A $2x + 4x^3 - 1$

Write terms in descending order by degree.

$4x^3 + 2x - 1$

Leading coefficient: 4

Degree: 3

Terms: 3

Name: cubic trinomial

B $7x^3 - 11x + x^5 - 2$

Write terms in descending order by degree.

$1x^5 + 7x^3 - 11x - 2$

Leading coefficient: 1

Degree: 5

Terms: 4

Name: quintic polynomial with four terms



Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

2a. $4x - 2x^2 + 2$

2b. $-18x^2 + x^3 - 5 + 2x$

To add or subtract polynomials, combine like terms. You can add or subtract horizontally or vertically.

EXAMPLE 3 Adding and Subtracting Polynomials

Add or subtract. Write your answer in standard form.

A $(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$

Add vertically.

$(3x^2 + 7 + x) + (14x^3 + 2 + x^2 - x)$

$3x^2 + x + 7$ Write in standard form.

$+ 14x^3 + x^2 - x + 2$ Align like terms.

$14x^3 + 4x^2 + 0x + 9$ Add.

$14x^3 + 4x^2 + 9$ Combine like terms.

Remember!

To subtract a polynomial, distribute the negative to all terms.

Add or subtract. Write your answer in standard form.

B $(1 - x^2) - (3x^2 + 2x - 5)$

Add the opposite horizontally.

$$(1 - x^2) - (3x^2 + 2x - 5)$$

$$(-x^2 + 1) + (-3x^2 - 2x + 5)$$

Write in standard form.

$$(-x^2 - 3x^2) + (-2x) + (1 + 5)$$

Group like terms.

$$-4x^2 - 2x + 6$$

Add.



Add or subtract. Write your answer in standard form.

3a. $(-36x^2 + 6x - 11) + (6x^2 + 16x^3 - 5)$

3b. $(5x^3 + 12 + 6x^2) - (15x^2 + 3x - 2)$

A **polynomial function** is a function whose rule is a polynomial. In this course, you will study only polynomial functions with one variable.

EXAMPLE 4 Medical Application

Cardiac output is the amount of blood pumped through the heart. The output is measured by a technique called dye dilution. A doctor injects dye into a vein near the heart and measures the amount of dye in the arteries over time.

The cardiac output of a particular patient can be approximated by the function $f(t) = 0.0056t^3 - 0.22t^2 + 2.33t$, where t represents time (in seconds after injection, $0 \leq t \leq 23$) and $f(t)$ represents the concentration of dye (in milligrams per liter).

- a. Evaluate $f(t)$ for $t = 0$ and $t = 3$.

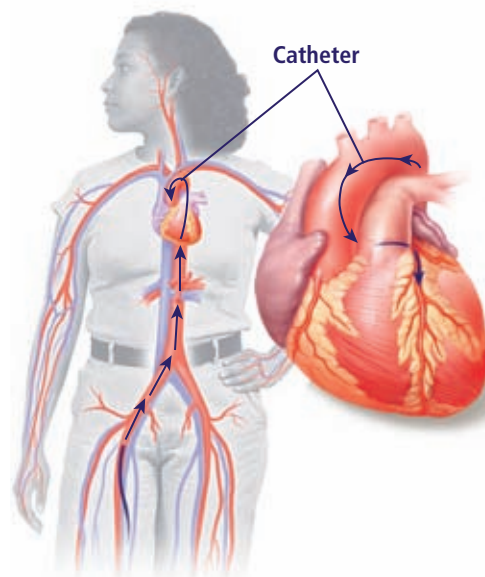
$$f(0) = 0.0056(0)^3 - 0.22(0)^2 + 2.33(0) = 0$$

$$f(3) = 0.0056(3)^3 - 0.22(3)^2 + 2.33(3) = 5.1612$$

- b. Describe what the values of the function from part a represent.

$f(0)$ represents the concentration of dye, 0 mg/L, in the artery at the start of the dye dilution process.

$f(3)$ represents the concentration of dye, 5.1612 mg/L, in the artery after 3 seconds.



4. For a different patient, the dye dilution can be modeled by the function $f(t) = 0.000468x^4 - 0.016x^3 + 0.095x^2 + 0.806x$. Evaluate $f(t)$ for $t = 4$ and $t = 17$, and describe what the values of the function represent.

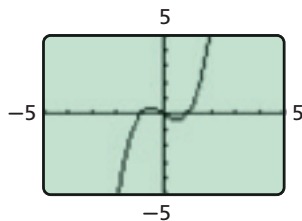
Graphing polynomial functions can be a challenge. Throughout this chapter, you will learn skills for analyzing, describing, and graphing higher-degree polynomials. Until then, the graphing calculator will be a useful tool.

EXAMPLE 5 Graphing Higher-Degree Polynomials on a Calculator

Caution!

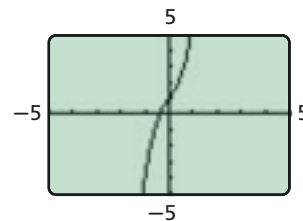
Depending on your viewing window, a calculator may not show all of the important features of a graph. Watch out for hidden behavior.

A $f(x) = x^3 - x$



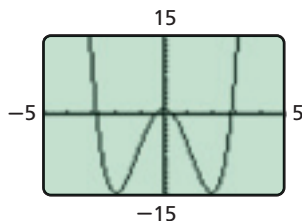
From left to right, the graph increases, decreases slightly, and then increases again. It crosses the x -axis three times, so there appear to be three real zeros.

B $f(x) = 3x^3 + 2x + 1$



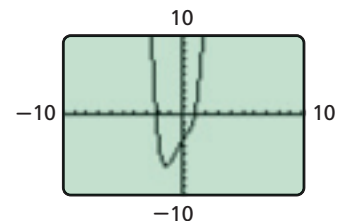
From left to right, the graph increases. It crosses the x -axis once, so there appears to be one real zero.

C $h(x) = x^4 - 8x^2 + 1$



From left to right, the graph alternately decreases and increases, changing direction three times. It crosses the x -axis four times, so there appear to be four real zeros.

D $k(x) = x^4 + x^3 - x^2 + 2x - 3$



From left to right, the graph decreases and then increases. It crosses the x -axis twice, so there appear to be two real zeros.



Graph each polynomial on a calculator. Describe the graph, and identify the number of real zeros.

5a. $f(x) = 6x^3 + x^2 - 5x + 1$

5b. $f(x) = 3x^2 - 2x + 2$

5c. $g(x) = x^4 - 3$

5d. $h(x) = 4x^4 - 16x^2 + 5$

THINK AND DISCUSS

- Can a polynomial have a leading coefficient of $\sqrt{3}$? Explain.
- What is the degree of the sum of a quartic polynomial and a cubic polynomial? Explain.
- Is the sum of two trinomial polynomials always a trinomial? Explain.
- GET ORGANIZED** Copy and complete the graphic organizer.

Characteristics	Definition
Polynomial	
Examples	Nonexamples





GUIDED PRACTICE

1. **Vocabulary** Explain how to identify the leading coefficient of a polynomial.

SEE EXAMPLE 1 Identify the degree of each monomial.

p. 406

2. $-7x$

3. $4x^2y^3$

4. 13

5. m^3n^2p

SEE EXAMPLE 2 Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

p. 407

6. $4x + 2x^2 - 7 + x^3$

7. $3x^2 + 5x - 4$

8. $5x^2 - 4x^3$

9. $4x^4 + 8x^2 + 1 - 3x$

SEE EXAMPLE 3 Add or subtract. Write your answer in standard form.

p. 407

10. $(15x^2 - 3x + 11) + (2x^3 - x^2 + 6x + 1)$

11. $(12x - 1 + 2x^2) + (x^2 + 4)$

12. $(3x^2 - 5x) - (-4 + x^2 + x)$

13. $(x^2 - 3x + 7) - (6x^2 + 4x + 12)$

SEE EXAMPLE 4 14. **Number Theory** The sum of the squares of the first n natural numbers is given by the polynomial function $F(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

p. 408

a. Evaluate $F(n)$ for $n = 5$ and $n = 10$.

b. Describe what the values of the function from part a represent.

SEE EXAMPLE 5 Graph each polynomial function on a calculator. Describe the graph, and identify the number of real zeros.

p. 409

15. $f(x) = 4x^3 + 2x + 1$

16. $g(x) = \frac{1}{4}x^4 - 3x^2$

17. $h(x) = -3x^3 - 6$

18. $p(x) = -4x^4 + 6x^3 - 3x^2$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
19–22	1
23–26	2
27–30	3
31	4
32–35	5

Extra Practice

Skills Practice p. S14

Application Practice p. S37

Identify the degree of each monomial.

19. x^8

20. $6x^3y$

21. 8

22. $a^4b^6c^3$

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

23. $3x^3 + 2x^4 - 7x + x^2$

24. $6x - 4x^4 + 5^7$

25. $2x^3 + 10x - 9$

26. $3x^2 + 2x^6 - 4x^4 - 1$

Add or subtract. Write your answer in standard form.

27. $(x^2 - 3x + 4) + (x^3 + 3x - 4)$

28. $(x^2 - 3x + 4) - (3x + x^3 - 4)$

29. $(5y^3 - 2y^2 - 1) - (y^2 - 2y - 3)$

30. $(2y^2 - 5y + 3) + (y^2 - 2y - 5)$

31. **Recreation** The distance d , in centimeters, that a diving board bends below its resting position when you stand at its end is dependent on your distance x , in meters, from the stabilized point. This relationship can be modeled by the function $d(x) = -4x^3 + x^2$.

a. Evaluate $d(x)$ for $x = 1$ and $x = 2$.

b. Describe what the values of the function from part a represent.

Graph each polynomial function on a calculator. Describe the graph, and identify the number of real zeros.

32. $f(x) = -2x^2 + x - 1$

33. $g(x) = x^3 + 1$

34. $h(x) = x^4 - 6x^2 + 10$

35. $p(x) = -x^5 + x - 1$

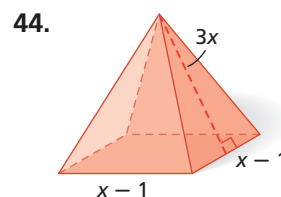
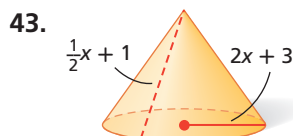
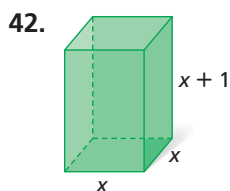
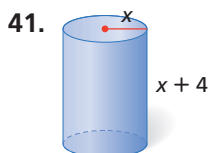
Complete the table.

	Polynomial	Standard Form	Leading Coefficient	Degree
36.	$8x + 3x^2 - 5$	☐	☐	☐
37.	$3x^2 + x^4 - 2$	☐	☐	☐
38.	$x^3 - x^4 + x - 1$	☐	☐	☐
39.	$64 + x^2$	☐	☐	☐

40. **Critical Thinking** Write a quartic trinomial with a leading coefficient of 2.



Geometry Find a polynomial expression in terms of x for the surface area of each figure.



45. **Business** The manager of a gift-basket business will ship the baskets anywhere in the country. The cost to mail a basket based on its weight x , in pounds, is given by $C(x) = 0.03x^3 - 0.75x^2 + 4.5x + 7$.

- What is the cost of shipping a 7-pound gift basket?
- What is the cost of shipping a 19-pound gift basket?

46. **Estimation** Estimate the value of $P(x) = -2.03x^3 + \pi x^2 - x + 5.8$ for $P(-2.78)$.

Tell whether each statement is sometimes, always, or never true. If it is sometimes true, give examples to support your answer.

- A quadratic polynomial is a trinomial.
- The degree of a polynomial in standard form is equal to the degree of the first term.
- The leading coefficient of a polynomial is the greatest coefficient of any term.

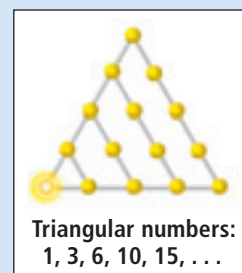
**MULTI-STEP
TEST PREP**



50. This problem will prepare you for the Multi-Step Test Prep on page 436.

The total number of lights in a triangular lighting rig is related to the triangular numbers, as shown at right. The n th triangular number is given by $T(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

- Write a polynomial function that represents the $(n + 1)$ th triangular number, $T(n + 1)$.
- The difference between two consecutive triangular numbers is $T(n + 1) - T(n)$. Subtract these two polynomial functions, and state a conclusion about the difference between consecutive triangular numbers.





51. Graphing Calculator The functions below are polynomials in factored form. Graph each function. Identify the x -intercepts. What can you say about the x -intercepts and the linear binomial factors in the functions?

- a. $f(x) = (x + 3)(x - 1)(x - 4)$
- b. $g(x) = (x + 1)(x + 2)(x - 3)(x - 1)$
- c. $h(x) = x(x + 1)(x - 2)$
- d. $k(x) = (x + 2)(x - 3)$
- e. $j(x) = x\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$



Write About It Recall the properties of real numbers from Lesson 1-2.

- 52. Is the addition of polynomial functions commutative? Explain.
- 53. Is the addition of polynomial functions associative? Explain.



54. What is the degree of the monomial $5xy^4z$?

- (A) 6
- (B) 1
- (C) 4
- (D) 5

55. For $f(x) = 2x^2 + 4x - 6$ and $g(x) = 2x^2 + 2x + 8$, find $f(x) - g(x)$.

- (F) $-4x^2 - 2x + 2$
- (G) $2x + 2$
- (H) $4x^2 + 6x + 2$
- (J) $2x - 14$

56. Which polynomial is written in standard form?

- (A) $7 + 2x^4 - x^6$
- (B) $3x^3 - x^5$
- (C) x^4
- (D) $x^2 + 3 - 2x$

57. What is the degree of the polynomial function $h(x) = 7x^3 - x^6 + x$?

- (F) 10
- (G) 3
- (H) -1
- (J) 6

58. **Short Response** Evaluate $P(x) = \frac{1}{2}x^3 - x^2 + 8$ for $x = -2$.

CHALLENGE AND EXTEND

$P(x)$ and $R(x)$ are polynomials. $P(x)$ is a trinomial. Give examples of $P(x)$ and $R(x)$ that meet the given conditions.

- 59. $P(x) - R(x)$ is a binomial.
- 60. $P(x) - R(x)$ is a trinomial.
- 61. $P(x) - R(x)$ is a polynomial with four terms.
- 62. $P(x) - R(x)$ is a quartic.
- 63. $P(x) - R(x)$ is a quintic.

SPIRAL REVIEW

Graph each line. (Lesson 2-3)

- 64. slope $\frac{3}{4}$, point $(0, -1)$
- 65. slope -2 , point $(3, 0)$
- 66. slope 1, point $(1, 2)$

Determine if each line is vertical or horizontal. Then graph the line. (Lesson 2-3)

- 67. $x = 4$
- 68. $y = -2$
- 69. $y = \frac{3}{4}$

Using $f(x) = x^2$ as a guide, graph each function and describe the transformations. (Lesson 5-1)

- 70. $g(x) = (x - 5)^2 + 6$
- 71. $g(x) = (x + 3)^2 + 2$
- 72. $h(x) = \frac{1}{5}x^2 + 2$

Pascal's Triangle

Each number in Pascal's triangle is the sum of the two numbers diagonally above it. All of the outside numbers are 1.

Many interesting number patterns can be found in Pascal's triangle, such as Fibonacci's sequence and powers of 2.

Pascal's Triangle is useful for many different mathematical situations, such as expanding binomials and probability.

[illegible]

Activity

Find rows 6 and 7 of Pascal's triangle.

Row 5 → 1 5 10 10 5 1

Row 6 → **1** **6** **15** **20** **15** **6** **1**

Row 7 → **1** **7** **21** **35** **35** **21** **7** **1**

All of the outside numbers are 1. Fill in values by adding the numbers in row 5 that are diagonally above the new values.

Repeat the process for row 7.

Try This

- Find rows 8, 9, and 10 of Pascal's triangle.
- Make a Conjecture** What can you say about the relationship between the row number and the number of terms in a row?
- Make a Conjecture** What can you say about the relationship between the row number and the second term in each row?
- Make a Conjecture** Expand $(x + 1)(x + 1)$ and $(x + 1)(x + 1)(x + 1)$, and use your answers to make a conjecture about the relationship between Pascal's triangle and the multiplication of binomials.
- Test your conjecture from Problem 4 by expanding $(x + 1)(x + 1)(x + 1)(x + 1)$ with multiplication and by using Pascal's triangle.



6-2

Multiplying Polynomials

Objectives

Multiply polynomials.

Use binomial expansion to expand binomial expressions that are raised to positive integer powers.

Who uses this?

Business managers can multiply polynomials when modeling total manufacturing costs. (See Example 3.)

To multiply a polynomial by a monomial, use the Distributive Property and the Properties of Exponents.



EXAMPLE 1 Multiplying a Monomial and a Polynomial

Remember!

To review Properties of Exponents, refer to Lesson 1-5.

Find each product.

A $3x^2(x^3 + 4)$

$$3x^2(x^3 + 4)$$

$$3x^2 \cdot x^3 + 3x^2 \cdot 4 \quad \text{Distribute.}$$

$$3x^5 + 12x^2 \quad \text{Multiply.}$$

B $ab(a^3 + 3ab^2 - b^3)$

$$ab(a^3 + 3ab^2 - b^3)$$

$$ab(a^3) + ab(3ab^2) + ab(-b^3)$$

$$a^4b + 3a^2b^3 - ab^4$$



Find each product.

1a. $3cd^2(4c^2d - 6cd + 14cd^2)$ **1b.** $x^2y(6y^3 + y^2 - 28y + 30)$

To multiply any two polynomials, use the Distributive Property and multiply each term in the second polynomial by each term in the first.

$$(x + 2)(x^2 + 4x - 3)$$

Keep in mind that if one polynomial has m terms and the other has n terms, then the product has mn terms before it is simplified.

EXAMPLE 2 Multiplying Polynomials

Find each product.

A $(x - 2)(1 + 3x - x^2)$

Method 1 Multiply horizontally.

$$(x - 2)(-x^2 + 3x + 1)$$

$$x(-x^2) + x(3x) + x(1) - 2(-x^2) - 2(3x) - 2(1) \quad \text{Write polynomials in standard form.}$$

$$-x^3 + 3x^2 + x + 2x^2 - 6x - 2 \quad \text{Distribute } x \text{ and then } -2.$$

$$-x^3 + 5x^2 - 5x - 2 \quad \text{Multiply. Add exponents.}$$

Combine like terms.

Helpful Hint

When using a table to multiply, the polynomials must be in standard form. Use a zero for any missing terms.

Method 2 Multiply vertically.

$$\begin{array}{r}
 -x^2 + 3x + 1 \\
 \times x - 2 \\
 \hline
 -x^3 + 3x^2 + x \\
 -x^3 + 5x^2 - 5x - 2 \\
 \hline
 \end{array}$$

Write each polynomial in standard form.

Multiply $(-x^2 + 3x + 1)$ by -2 .

Multiply $(-x^2 + 3x + 1)$ by x , and align like terms.

Combine like terms.

Find each product.

B $(x^2 + 3x - 5)(x^2 - x + 1)$

Multiply each term of one polynomial by each term of the other. Use a table to organize the products.

	x^2	$-x$	$+1$
x^2	x^4	$-x^3$	$+x^2$
$+3x$	$+3x^3$	$-3x^2$	$+3x$
-5	$-5x^2$	$+5x$	-5

The top left corner is the first term in the product. Combine terms along diagonals to get the middle terms. The bottom right corner is the last term in the product.

$$x^4 + (3x^3 - x^3) + (-5x^2 - 3x^2 + x^2) + (5x + 3x) + (-5)$$

$$x^4 + 2x^3 - 7x^2 + 8x - 5$$



Find each product.

2a. $(3b - 2c)(3b^2 - bc - 2c^2)$ **2b.** $(x^2 - 4x + 1)(x^2 + 5x - 2)$

EXAMPLE 3 Business Application

Mr. Silva manages a manufacturing plant. From 1990 through 2005, the number of units produced (in thousands) can be modeled by $N(x) = 0.02x^2 + 0.2x + 3$. The average cost per unit (in dollars) can be modeled by $C(x) = -0.002x^2 - 0.1x + 2$, where x is the number of years since 1990. Write a polynomial $T(x)$ that can be used to model Mr. Silva's total manufacturing costs.

Total cost is the product of the number of units and the cost per unit.

$$T(x) = N(x) \cdot C(x).$$

Multiply the two polynomials.

$$\begin{array}{r}
 0.02x^2 + 0.2x + 3 \\
 \times -0.002x^2 - 0.1x + 2 \\
 \hline
 0.04x^2 + 0.4x + 6 \\
 -0.002x^3 - 0.02x^2 - 0.3x \\
 -0.00004x^4 - 0.0004x^3 - 0.006x^2 \\
 \hline
 -0.00004x^4 - 0.0024x^3 + 0.014x^2 + 0.1x + 6
 \end{array}$$

Mr. Silva's total manufacturing costs, in thousands of dollars, can be modeled by $T(x) = -0.00004x^4 - 0.0024x^3 + 0.014x^2 + 0.1x + 6$.



3. What if...? Suppose that in 2005 the cost of raw materials increases and the new average cost per unit is modeled by $C(x) = -0.004x^2 - 0.1x + 3$. Write a polynomial $T(x)$ that can be used to model the total costs.

You can also raise polynomials to powers.

EXAMPLE 4 Expanding a Power of a Binomial

Find the product.

$$(x + y)^3$$

$$(x + y)(x + y)(x + y)$$

Write in expanded form.

$$(x + y)(x^2 + 2xy + y^2)$$

Multiply the last two binomial factors.

$$x(x^2) + x(2xy) + x(y^2) + y(x^2) + y(2xy) + y(y^2)$$

Distribute x and then y.

$$x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$

Multiply.

$$x^3 + 3x^2y + 3xy^2 + y^3$$

Combine like terms.



Find each product.

4a. $(x + 4)^4$

4b. $(2x - 1)^3$

Notice the coefficients of the variables in the final product of $(x + y)^3$. These coefficients are the numbers from the third row of Pascal's triangle.

Binomial Expansion	Pascal's Triangle (Coefficients)
$(a + b)^0 = 1$	1
$(a + b)^1 = a + b$	1 1
$(a + b)^2 = a^2 + 2ab + b^2$	1 2 1
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	1 5 10 10 5 1

Each row of Pascal's triangle gives the coefficients of the corresponding binomial expansion. The pattern in the table can be extended to apply to the expansion of any binomial of the form $(a + b)^n$, where n is a whole number.



Binomial Expansion

For a binomial expansion of the form $(a + b)^n$, the following statements are true.

1. There are $n + 1$ terms.
2. The coefficients are the numbers from the n th row of Pascal's triangle.
3. The exponent of a is n in the first term, and the exponent decreases by 1 in each successive term.
4. The exponent of b is 0 in the first term, and the exponent increases by 1 in each successive term.
5. The sum of the exponents in any term is n .

This information is formalized by the *Binomial Theorem*, which you will study further in Chapter 11.

Student to Student

Expanding Binomials



Caitlin Humphrey
Hillcrest High School

I like to use a chart to expand binomials. I will use the binomial $(x + 2)^4$ as an example.

I write the coefficients from Pascal's triangle in the top row. I write the decreasing powers in the second row.

1	4	6	4	1
x^4	x^3	x^2	x	
	2	4	8	16

Then I shift one column to the right and write the increasing powers in the third row.

Finally, I multiply vertically to get $x^4 + 8x^3 + 24x^2 + 32x + 16$.

EXAMPLE 5 Using Pascal's Triangle to Expand Binomial Expressions

Expand each expression.

A $(y - 3)^4$

1 4 6 4 1 Identify the coefficients for $n = 4$, or row 4.

$$[1y^4(-3)^0] + [4y^3(-3)^1] + [6y^2(-3)^2] + [4y^1(-3)^3] + [1y^0(-3)^4]$$

$$y^4 - 12y^3 + 54y^2 - 108y + 81$$

B $(4z + 5)^3$

1 3 3 1 Identify the coefficients for $n = 3$, or row 3.

$$[1(4z)^35^0] + [3(4z)^25^1] + [3(4z)^15^2] + [1(4z)^05^3]$$

$$64z^3 + 240z^2 + 300z + 125$$



Expand each expression.

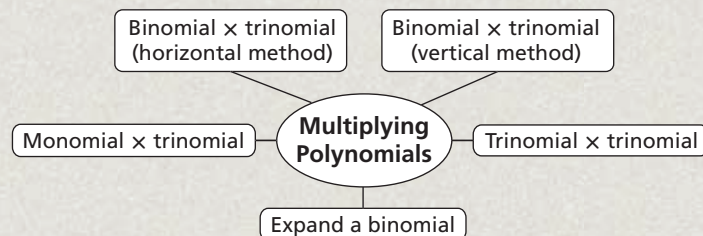
5a. $(x + 2)^3$

5b. $(x - 4)^5$

5c. $(3x + 1)^4$

THINK AND DISCUSS

- The product of $(3x^4 - 2x^2 - 1)$ and a polynomial $P(x)$ results in a polynomial of degree 9. What is the degree of $P(x)$? Explain.
- After $(2x + 8)^7$ is expanded, what is the degree of the result, and how many terms does the result have? Explain.
- GET ORGANIZED** Copy and complete the graphic organizer. In each box, write an example and find the product.





GUIDED PRACTICE

Find each product.

SEE EXAMPLE 1

p. 414

1. $-4c^2d^3(5cd^2 + 3c^2d)$

3. $xy(5x^2 + 8x - 7)$

2. $3x^2(2y + 5x)$

4. $2xy(3x^2 - xy + 7)$

SEE EXAMPLE 2

p. 414

5. $(x - y)(x^2 + 2xy - y^2)$

7. $(x^3 + 3x^2 + 1)(3x^2 + 6x - 2)$

6. $(3x - 2)(2x^2 + 3x - 1)$

8. $(x^2 + 9x + 7)(3x^2 + 9x + 5)$

SEE EXAMPLE 3

p. 415

9. **Business** A businessman models the number of items (in thousands) that his company sold from 1998 through 2004 as $N(x) = -0.1x^3 + x^2 - 3x + 4$ and the average price per item (in dollars) as $P(x) = 0.2x + 5$, where x represents the number of years since 1998. Write a polynomial $R(x)$ that can be used to model the total revenue for this company.

SEE EXAMPLE 4

p. 416

Find each product.

10. $(x + 2)^3$

11. $(x + y)^4$

12. $(x + 1)^4$

13. $(x - 3y)^3$

SEE EXAMPLE 5

p. 417

Expand each expression.

14. $(x - 2)^4$

15. $(2x + y)^4$

16. $(x + 2y)^3$

17. $(2x - y)^5$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
18–21	1
22–25	2
26	3
27–30	4
31–34	5

Extra Practice

Skills Practice p. S14

Application Practice p. S37

Find each product.

18. $7x^3(2x + 3)$

20. $xy^2(x^2 + 3xy + 9)$

22. $(x - y)(x^2 - xy + y^2)$

24. $(x^3 + x^2 + 1)(x^2 - x - 5)$

19. $3x^2(2x^2 + 9x - 6)$

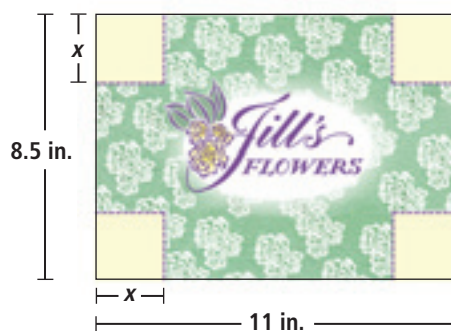
21. $2r^2(6r^3 + 14r^2 - 30r + 14)$

23. $(2x + 5y)(3x^2 - 4xy + 2y^2)$

25. $(4x^2 + 3x + 2)(3x^2 + 2x - 1)$

26. **Measurement** A bottom for a box can be made by cutting congruent squares from each of the four corners of a piece of cardboard. The volume of a box made from an 8.5-by-11-inch piece of cardboard would be represented by $V(x) = x(11 - 2x)(8.5 - 2x)$, where x is the side length of one square.

a. Express the volume as a sum of monomials.

b. Find the volume when $x = 1$ inch.

Find each product.

27. $(2x - 2)^3$

28. $\left(x + \frac{1}{3}\right)^4$

29. $(x - y)^4$

30. $(4 + y)^3$

Expand each expression.

31. $(x - 3y)^4$

32. $(x - 2)^5$

33. $(x + y)^5$

34. $(2x - 3y)^4$



Graphing Calculator Compare each pair of expressions with your graphing calculator. Use the table feature to make a conjecture about whether the expressions are equivalent.

35. $(x - 6)^3$; $x^3 - 18x^2 + 108x - 216$ 36. $(11x + 10)(11x + 1)$; $121x^2 + 121x + 10$

37. $(3x^2 + 2x)(3x + 2)$; $9x^3 + 12x^2 + 4$ 38. $(2x + 1)^4$; $16x^4 + 32x^3 + 24x^2 + 8x + 1$

39. **Business** Ms. Liao runs a small dress company. From 1995 through 2005, the number of dresses she made can be modeled by $N(x) = 0.3x^2 - 1.6x + 14$ and the average cost to make each dress can be modeled by $C(x) = -0.001x^2 - 0.06x + 8.3$, where x is the number of years since 1995. Write a polynomial that can be used to model Ms. Liao's total dressmaking costs, $T(x)$, for those years.



Math History



The Binomial Theorem was discovered by Sir Isaac Newton in 1665 or 1666. At this time, Trinity College, in Cambridge, England, was closed because of the plague, and Newton, working in solitude, was able to focus solely on his mathematical studies.

Multiply.

40. $-6x^3(15y^4 - 7xy^3 + 2)$

41. $(p - 2q)^3$

42. $(x^2 - 2yz - y^2)(y^2 + x)$

43. $(x^4 + xy^3)(x^2 + y^3)$

44. $(3 - 3y)^4$

45. $(5x^3 + x^2 - 9x)(y + 2)$

46. $(3 + x - 2x^2)(x - 1)$

47. $3(x - 2)^4$

48. $(x - 6)(x^4 - 2x^3 + x^2 + 1)$

49. $(30 + x^3 + x^2)(x - 15 - x^2)$

50. $\left(\frac{1}{2} + z\right)^4$

51. $(2x - 3)(x^5 - 4x^3 + 7)$

52. Generate the coefficients that would be used to expand $(a + b)^7$ by using binomial expansion.

53. **Physics** An object t seconds after it is thrown in the air has a velocity that can be described by $v(t) = -9.8t + 24$ (in meters/second) and a height $h(t) = -4.9t^2 + 24t + 60$ (in meters). The object has mass $m = 2$ kilograms. The kinetic energy of the object is given by $K = \frac{1}{2}mv^2$, and the potential energy is given by $U = 9.8mh$. Can you find a polynomial expression for the total kinetic and potential energy $K + U$ as a function of time, t ? Explain.

54. **ERROR ANALYSIS** Two students used binomial expansion to expand $(a + b)^2$. Which answer is incorrect? Identify the error.

A

$(a + b)^2$
$1a^2b^0 + 2a^1b^1 + 1a^0b^2$
$a^2 + 2ab + b^2$

B

$(a + b)^2$
$1a^2b^2 + 2a^1b^1 + 1a^0b^0$
$a^2b^2 + 2ab + 1$

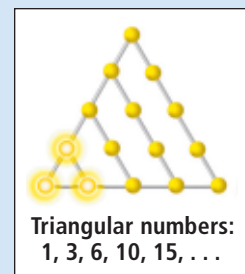
MULTI-STEP TEST PREP



55. This problem will prepare you for the Multi-Step Test Prep on page 436.

The total number of lights in a triangular lighting rig is related to the triangular numbers, as shown at right. The product of the n th triangular number and the $(n + 1)$ th triangular number is given by $f(n) = \frac{n(n+1)^2(n+2)}{4}$.

- Write $f(n)$ as a polynomial function.
- Find the product of the twelfth and thirteenth triangular numbers.
- Evaluate $f(n)$ for $n = 20$ and describe what this value represents.



56. **Critical Thinking** Using binomial expansion, explain why every other term of the resulting polynomial for $(x - y)^5$ is negative.



57. **Write About It** Explain how to expand a binomial raised to a power by using Pascal's Triangle.



58. Multiply $(y - 3)(y^2 - 6y - 9)$.

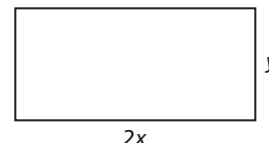
(A) $y^3 + 18y - 9$

(B) $y^3 - 3y^2 + 3y + 27$

(C) $y^3 + 9y^2 + 27y + 27$

(D) $y^3 - 9y^2 + 9y + 27$

59. The rectangle shown is enlarged such that each side is multiplied by the value of the width, $2x$. Which expression represents the perimeter of the enlarged rectangle?



(F) $4x + 2y$

(G) $6x + 4xy$

(H) $8x^2 + 2y$

(J) $8x^2 + 4xy$

60. What is the third term of the binomial expansion of $(x - 4)^6$?

(A) $240x^4$

(B) $15x^4$

(C) $160x^3$

(D) $8x^3$

61. Find the product $a^2b(2a^3b - 5ab^4)$.

(F) $-3a^4b^{-2}$

(G) $2a^6b - 5a^2b^4$

(H) $2a^5b^2 - 5a^3b^5$

(J) $2a^5b^2 - 5ab^4$

62. **Short Response** Expand $(4 - x)^4$ by using binomial expansion.

CHALLENGE AND EXTEND

Find the product.

63. $(x - 1)^{10}$

64. $(14 + y)^5$

65. $(m - n)^3(m + n)^3$

66. $(ab + 2c)^4$

Suppose $P(x) = x + 3$. Find a binomial $B(x)$ that satisfies the given condition.

67. $P(x) \cdot B(x)$ is a binomial.

68. $P(x) \cdot B(x)$ is a trinomial.

69. $P(x) \cdot B(x)$ is a quartic polynomial.

SPIRAL REVIEW

70. **Athletics** A basketball coach makes his players run five sprints for every point they lost by in a game. Write a function to represent the number of sprints the team has to run after losing a game. How many sprints must the players run if they lose a game by a score of 84–73? ([Lesson 1-7](#))

Use the following matrices for Exercises 71–74. Evaluate, if possible. ([Lesson 4-2](#))

$$A = \begin{bmatrix} -2 & 1 \\ 4 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 4 & 2 \\ 2 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 3 \\ -1 & 5 \\ 0 & 7 \end{bmatrix}$$

71. A^2

72. CA

73. B^2

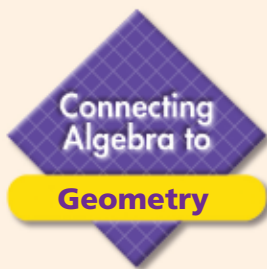
74. BC

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial. ([Lesson 6-1](#))

75. $3x + 5x^2 + 4x^4 - 6x^3$

76. $10x^2 + 5x^3 - x$

77. $9 - 4x^2 + 3x^5 - 2x$



See Skills Bank
page S65

Nets

For a prism, volume equals the area of the base times the height. For a pyramid, volume equals $\frac{1}{3}$ the area of the base times the height. To find the surface area of a solid, add the areas of all of the faces.

Activity

Find the volume and surface area of the square pyramid shown by this net.

For the volume, multiply the area of the square base by the height and then multiply by $\frac{1}{3}$.

$$B = (x - 3)^2$$

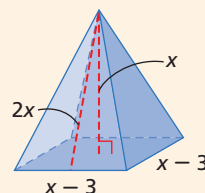
Find the area of the square base.

$$V = \frac{1}{3}(x - 3)^2 x$$

The height of the pyramid is x .

$$V = \frac{x^3}{3} - 2x^2 + 3x$$

Multiply the polynomials, and simplify.



For the surface area, add the area of the square base to the area of the four triangular faces.

$$B = (x - 3)^2$$

Find the area of the square base.

$$A = \frac{1}{2}(x - 3) \cdot 2x$$

Find the area of 1 triangular face.

$$L = 4A = 4 \cdot \frac{1}{2}(x - 3) \cdot 2x$$

Find the area of 4 triangular faces.

$$SA = (x - 3)^2 + 4 \cdot \frac{1}{2}(x - 3) \cdot 2x$$

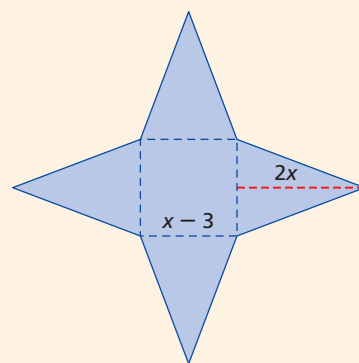
Add the area of the base to the area of the four triangular faces.

$$SA = (x^2 - 6x + 9) + (4x^2 - 12x)$$

Multiply.

$$SA = 5x^2 - 18x + 9$$

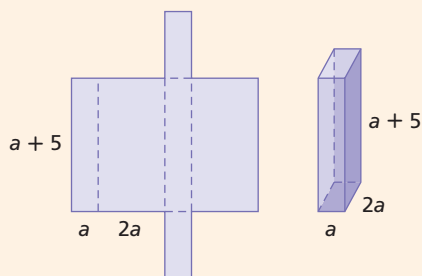
Add.



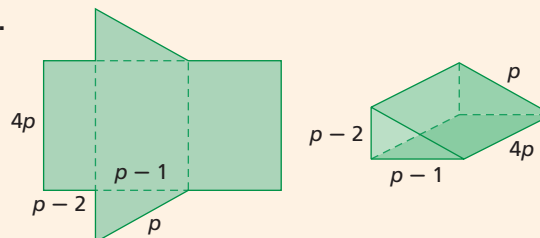
Try This

Find the volume and surface area of the solid shown by each net.

1.



2.



3. The volume of a rectangular prism is $6c^3 - 22c^2 - 8c$. Find the length and width of this prism if the height is $2c$.



6-3

Dividing Polynomials

Objective

Use long division and synthetic division to divide polynomials.

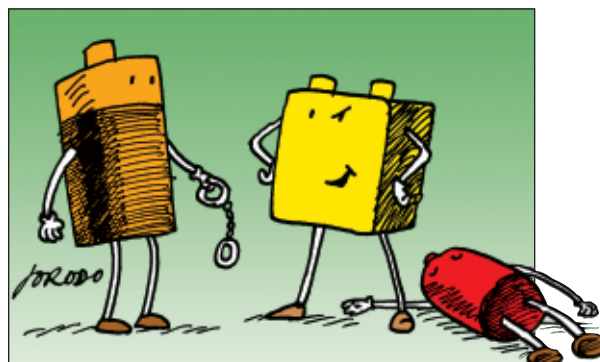
Vocabulary

synthetic division

Who uses this?

Electricians can divide polynomials in order to find the voltage in an electrical system. (See Example 4.)

Polynomial long division is a method for dividing a polynomial by another polynomial of a lower degree. It is very similar to dividing numbers.



© Cartoon Stock

Arithmetic Long Division

$$\begin{array}{r}
 \text{Divisor } 23 \leftarrow \text{Quotient} \\
 \text{Dividend } 12 \overline{)277} \leftarrow \text{Dividend} \\
 \underline{24} \\
 37 \\
 \underline{36} \\
 1 \leftarrow \text{Remainder}
 \end{array}$$

Polynomial Long Division

$$\begin{array}{r}
 \text{Divisor } x + 2 \leftarrow \text{Quotient} \\
 \text{Dividend } x + 2 \overline{)2x^2 + 7x + 7} \leftarrow \text{Dividend} \\
 \underline{2x^2 + 4x} \\
 3x + 7 \\
 \underline{3x + 6} \\
 1 \leftarrow \text{Remainder}
 \end{array}$$

EXAMPLE 1 Using Long Division to Divide Polynomials

Divide by using long division.

$$(4x^2 + 3x^3 + 10) \div (x - 2)$$

Step 1 Write the dividend in standard form, including terms with a coefficient of 0.

$$3x^3 + 4x^2 + 0x + 10$$

Step 2 Write division in the same way as you would when dividing numbers.

$$x - 2 \overline{)3x^3 + 4x^2 + 0x + 10}$$

Step 3 Divide.

$$\begin{array}{r}
 3x^2 + 10x + 20 \\
 x - 2 \overline{)3x^3 + 4x^2 + 0x + 10} \\
 \underline{-(3x^3 - 6x^2)} \\
 10x^2 + 0x \\
 \underline{-(10x^2 - 20x)} \\
 20x + 10 \\
 \underline{-(20x - 40)} \\
 50
 \end{array}$$

Notice that x times $3x^2$ is $3x^3$. Write $3x^2$ above $3x^3$.

Multiply $x - 2$ by $3x^2$. Then subtract.

Bring down the next term. Divide $10x^2$ by x .

Multiply $x - 2$ by $10x$, then subtract.

Bring down the next term. Divide $20x$ by x .

Multiply $x - 2$ by 20 , then subtract.

Find the remainder.

Step 4 Write the final answer.

$$\frac{4x^2 + 3x^3 + 10}{x - 2} = 3x^2 + 10x + 20 + \frac{50}{x - 2}$$



Divide by using long division.

$$1a. (15x^2 + 8x - 12) \div (3x + 1) \quad 1b. (x^2 + 5x - 28) \div (x - 3)$$

Synthetic division is a shorthand method of dividing a polynomial by a linear binomial by using only the coefficients. For synthetic division to work, the polynomial must be written in standard form, using 0 as a coefficient for any missing terms, and the divisor must be in the form $(x - a)$.



Synthetic Division Method

Divide $(2x^2 + 7x + 9) \div (x + 2)$ by using synthetic division.

WORDS	NUMBERS
Step 1 Write the coefficients of the dividend, 2, 7, and 9. In the upper left corner, write the value of a for the divisor $(x - a)$. So $a = -2$. Copy the first coefficient in the dividend below the horizontal bar.	$\begin{array}{r rrr} -2 & 2 & 7 & 9 \\ & \downarrow & & \\ & 2 & & \end{array}$
Step 2 Multiply the first coefficient by the divisor, and write the product under the next coefficient. Add the numbers in the new column.	$\begin{array}{r rrr} -2 & 2 & 7 & 9 \\ & \searrow & -4 & \\ & 2 & 3 & \end{array}$
Repeat Step 2 until additions have been completed in all columns. Draw a box around the last sum.	$\begin{array}{r rrr} -2 & 2 & 7 & 9 \\ & \searrow & -4 & -6 \\ & 2 & 3 & \boxed{3} \end{array}$
Step 3 The quotient is represented by the numbers below the horizontal bar. The boxed number is the remainder . The others are the coefficients of the polynomial quotient, in order of decreasing degree.	$= 2x + 3 + \frac{3}{x + 2}$

EXAMPLE 2 Using Synthetic Division to Divide by a Linear Binomial

Divide by using synthetic division.

A $(4x^2 - 12x + 9) \div \left(x + \frac{1}{2}\right)$

Step 1 Find a . Then write the coefficients and a in the synthetic division format.

$$a = -\frac{1}{2}$$

For $\left(x + \frac{1}{2}\right)$, $a = -\frac{1}{2}$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 4 & -12 & 9 & \\ & & & & \end{array}$$

Write the coefficients of $4x^2 - 12x + 9$.

Step 2 Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 4 & -12 & 9 & \\ & & -2 & 7 & \\ \hline & 4 & -14 & \boxed{16} & \end{array}$$

Draw a box around the remainder, 16.

Step 3 Write the quotient.

$$4x - 14 + \frac{16}{x + \frac{1}{2}}$$

Write the remainder over the divisor.

Check Multiply $\left(x + \frac{1}{2}\right)\left(4x - 14 + \frac{16}{x + \frac{1}{2}}\right)$.

$$4x\left(x + \frac{1}{2}\right) - 14\left(x + \frac{1}{2}\right) + \frac{16}{x + \frac{1}{2}}\left(x + \frac{1}{2}\right) = 4x^2 - 12x + 9 \quad \checkmark$$

Caution!

Be careful to use the correct a value when doing synthetic division. If the divisor is $(x - a)$, use a . If the divisor is $(x + a)$, use $-a$.

Divide by using synthetic division.

B $(x^4 - 2x^3 + 3x + 1) \div (x - 3)$

Step 1 Find a .

$$a = 3$$

For $(x - 3)$, $a = 3$.

Step 2 Write the coefficients and a in the synthetic division format.

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & 0 & 3 & 1 \end{array}$$

Use 0 for the coefficient of x^2 .

Step 3 Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & 0 & 3 & 1 \\ & & 3 & 3 & 9 & 36 \\ \hline & 1 & 1 & 3 & 12 & \boxed{37} \end{array}$$

Draw a box around the remainder, 37.

Step 4 Write the quotient.

$$x^3 + x^2 + 3x + 12 + \frac{37}{x-3}$$

Write the remainder over the divisor.



Divide by using synthetic division.

2a. $(6x^2 - 5x - 6) \div (x + 3)$ **2b.** $(x^2 - 3x - 18) \div (x - 6)$

You can use synthetic division to evaluate polynomials. This process is called synthetic substitution. The process of synthetic substitution is exactly the same as the process of synthetic division, but the final answer is interpreted differently, as described by the Remainder Theorem.



Remainder Theorem

THEOREM	EXAMPLE
If the polynomial function $P(x)$ is divided by $x - a$, then the remainder r is $P(a)$.	Divide $x^3 - 4x^2 + 5x + 1$ by $x - 3$. $\begin{array}{r rrrr} 3 & 1 & -4 & 5 & 1 \\ & & 3 & -3 & 6 \\ \hline & 1 & -1 & 2 & \boxed{7} \end{array}$ $P(3) = 7$

EXAMPLE 3 Using Synthetic Substitution

Use synthetic substitution to evaluate the polynomial for the given value.

A $P(x) = x^3 - 4x^2 + 3x - 5$ for $x = 4$

$$\begin{array}{r|rrrr} 4 & 1 & -4 & 3 & -5 \\ & & 4 & 0 & 12 \\ \hline & 1 & 0 & 3 & \boxed{7} \end{array}$$

Write the coefficients of the dividend.

Use $a = 4$.

$$P(4) = 7$$

Check Substitute 4 for x in $P(x) = x^3 - 4x^2 + 3x - 5$.

$$P(4) = 4^3 - 4(4)^2 + 3(4) - 5$$

$$P(4) = 64 - 64 + 12 - 5$$

$$P(4) = 7 \checkmark$$

Use synthetic substitution to evaluate the polynomial for the given value.

B $P(x) = 4x^4 + 2x^3 + 3x + 5$ for $x = -\frac{1}{2}$

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 4 & 2 & 0 & 3 & 5 \\ & & -2 & 0 & 0 & -\frac{3}{2} \\ \hline & 4 & 0 & 0 & 3 & 3\frac{1}{2} \end{array}$$

Write the coefficients of the dividend.

Use 0 for the coefficient of x^2 and $a = -\frac{1}{2}$.

$$P\left(-\frac{1}{2}\right) = 3\frac{1}{2}$$

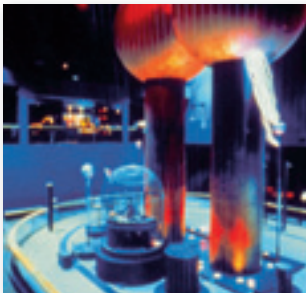


Use synthetic substitution to evaluate the polynomial for the given value.

3a. $P(x) = x^3 + 3x^2 + 4$ for $x = -3$

3b. $P(x) = 5x^2 + 9x + 3$ for $x = \frac{1}{5}$

EXAMPLE 4 Physics Application



A Van de Graaff generator is a machine that produces very high voltages by using small, safe levels of electric current. One machine has a current that can be modeled by $I(t) = t + 2$, where $t > 0$ represents time in seconds. The power of the system can be modeled by $P(t) = 0.5t^3 + 6t^2 + 10t$. Write an expression that represents the voltage of the system.

The voltage V is related to current I and power P by the equation $V = \frac{P}{I}$.

$$V(t) = \frac{0.5t^3 + 6t^2 + 10t}{t + 2}$$

Substitute.

$$\begin{array}{r|rrrr} -2 & 0.5 & 6 & 10 & 0 \\ & & -1 & -10 & 0 \\ \hline & 0.5 & 5 & 0 & 0 \end{array}$$

Use synthetic division.

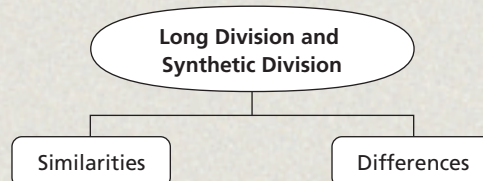
The voltage can be represented by $V(t) = 0.5t^2 + 5t$.



- 4.** Write an expression for the length of a rectangle with width $y - 9$ and area $y^2 - 14y + 45$.

THINK AND DISCUSS

- Can you use synthetic division to divide a polynomial by $x^2 + 3$? Explain.
- Explain how to quickly find $P(6)$ for the function $P(x) = 2x^3 - 11x^2 - 5x + 2$ without using a calculator. Find $P(6)$.
- GET ORGANIZED** Copy and complete the graphic organizer.





Geology



Bingham Canyon copper mine in Utah is the largest copper mine in the world in terms of both total metal production and size. Nicknamed “the richest hole on Earth,” the mine has produced more than 14.5 million tons of copper.

Fill in each box to illustrate the Remainder Theorem for $P(x) = x^2 + 3x - 7$ and divisor $x - 2$.

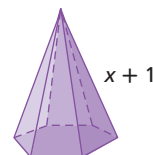
34. Divide $P(x)$ by $(x - 2)$: $\frac{P(x)}{x - 2} = \boxed{}$

35. Multiply both sides by $\boxed{}$: $P(x) = (x + 5)(x - 2) + 3$

36. Evaluate $P(2)$: $P(2) = \boxed{}$

37. **Geology** Geologists have taken a collection of samples of a substance from a proposed mining site and must identify the substance. Each sample is roughly cylindrical, and the volume of each sample as a function of cylinder height (in centimeters) is $V(h) = \frac{1}{4}\pi h^3$. The mass (in grams) of each sample in terms of height can be modeled by $M(h) = \frac{1}{4}h^3 - h^2 + 5h$. Write an expression that represents the density of the samples. (Hint: $D = \frac{M}{V}$)

38. **Geometry** The volume of a hexagonal pyramid is modeled by the function $V(x) = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{2}{3}x - \frac{1}{3}$. Use polynomial division to find an expression for the area of the base. (Hint: For a pyramid, $V = \frac{1}{3}Bh$.)



Divide.

39. $(y^4 + 9y^2 + 20) \div (y^2 + 4)$

40. $(2x^2 - 5x + 2) \div (x - \frac{1}{2})$

41. $(3x^3 - 11x^2 - 56x - 48) \div (3x + 4)$

42. $(60 - 16y^2 + y^4) \div (10 - y^2)$

43. $(t^3 - 7t^2 + 12t) \div (t^2 - 3t)$

44. $(y^2 - 18y + 14) \div (y - 1)$

45. $(x^4 - 3x^3 - 28x^2 + 59x + 6) \div (x - 6)$

46. $(2d^2 + 10d + 8) \div (2d + 2)$

47. $(x^4 - 7x^3 + 9x^2 - 22x + 25) \div (x - 6)$

48. $(6x^3 - 14x^2 + 10x - 4) \div (x - 1)$

49. **ERROR ANALYSIS** Two students used synthetic division to divide $x^3 - 2x - 8$ by $x - 2$. Determine which solution is correct. Find the error in the other solution.

A

2	1	0	-2	-8
		2	-4	4
	1	-2	2	-12

B

2	1	0	-2	-8
		2	4	4
	1	2	2	-4

50. **Critical Thinking** Is $x + 3$ a factor of $3x^3 + 5x^2 + 2x - 12$? Explain.



51. **Write About It** What conditions must be met in order to use synthetic division?

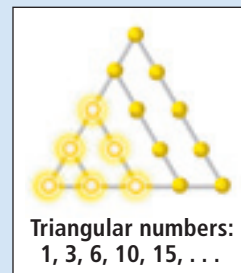
MULTI-STEP TEST PREP



52. This problem will prepare you for the Multi-Step Test Prep on page 436.

The total number of lights in a triangular lighting rig is related to the triangular numbers, as shown at right. The sum of the first n triangular numbers is given by the polynomial function $g(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$.

- Find the sum of the first five triangular numbers in the figure at right, and verify that the formula works when $n = 5$.
- Use synthetic substitution to find the sum of the first 24 triangular numbers.

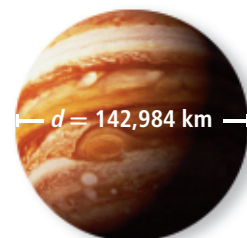


53. What is the remainder when $2x^2 + 6x + 3$ is divided by $x + 3$?
 (A) 39 (B) 3 (C) 1 (D) 0
54. Which expression is equivalent to $\frac{6a^2b + 9b^2}{3a^2}$?
 (F) $6b + \frac{9b^2}{a^2}$ (G) $\frac{3a^2}{6a^2b + 9b^2}$ (H) $2b + \frac{3b^2}{a^2}$ (J) $\frac{2a^2b + 3b^2}{3a^2}$
55. Which expression is equivalent to $(x^2 + 3x - 28) \div (x - 4)$?
 (A) $x + 7 + \frac{3}{x - 4}$ (B) $x - 7$ (C) $28 + \frac{4}{x - 7}$ (D) $x + 7$
56. **Gridded Response** Use synthetic substitution to evaluate $f(x) = 3x^4 - 6x^2 + 12$ for $x = -2$.

CHALLENGE AND EXTEND

Evaluate $P(x) = 4x^9 + 7x^7 - 6x^6 - 5x^4 - x^2 + 3x - 2$ for the given value of x .

57. $x = -4$ 58. $x = -1$ 59. $x = 1$ 60. $x = 3$
61. If -3 is a zero of $P(x) = 2x^3 + 3x^2 - kx - 27$, find the value of k .
62. Divide $(5a^2b - 3ab^2 - 2b^3)$ by $(ab - b^2)$.
63. **Astronomy** The volumes of several planets in cubic kilometers can be modeled by $V(d) = \frac{1}{6}\pi d^3$, where d is the diameter of the planet in kilometers. The mass of each planet in kilograms in terms of diameter d can be modeled by $M(d) = (3.96 \times 10^{12})d^3 - (6.50 \times 10^{17})d^2 + (2.56 \times 10^{22})d - 5.56 \times 10^{25}$.
- The density of a planet in kilograms per cubic kilometer can be found by dividing the planet's mass by its volume. Use polynomial division to find a model for the density of a planet in terms of its diameter.
 - Use the model to estimate the density of Jupiter.
 - Use the model to estimate the density of Neptune.



SPIRAL REVIEW

64. A class conducted a survey on eye color. The class found that 70% of students have brown eyes. If 448 students have brown eyes, how many students took the survey?
 (Lesson 2-2)

Find the maximum or minimum value of each function. Then state the domain and range of the function. (Lesson 5-2)

65. $f(x) = -4x^2 + 2x + 1$ 66. $g(x) = \frac{1}{2}x^2 - 5x + 6$
 67. $f(x) = \frac{1}{3}x^2 - 4x + 10$ 68. $g(x) = -\frac{1}{4}x^2 - 2x + 6$

Find each product. (Lesson 6-2)

69. $4x^2y(3xy^2 + 6x + 5y^3)$ 70. $5y^2(3xy + 4x^2y^2 - 8x^3y)$
 71. $(2x - 2y)(2x^2 - 2xy + 2y^2)$ 72. $2(y - 2)^4$



Explore the Sum and Difference of Two Cubes

You can use a diagram of a cube with a corner removed to discover how to factor the difference of two cubes. You can use a similar diagram to discover how to factor the sum of two cubes.

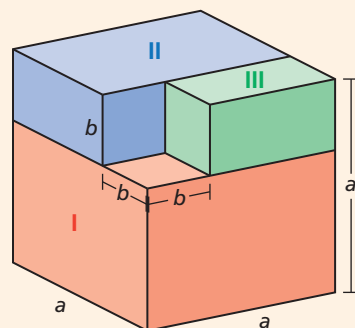
Use with Lesson 6-4

Activity

The figure shown is a large cube with a small cube removed from one corner.

- 1 Explain why the volume of the figure is $a^3 - b^3$. How is this related to the volumes of the rectangular prisms labeled I, II, and III?

The volume of the complete cube is a^3 , and the volume of the cube removed from the corner is b^3 . The volume of the figure with the corner removed is $a^3 - b^3$ and can be expressed as the sum of the volumes of the rectangular prisms I, II, and III: $a^3 - b^3 = V_I + V_{II} + V_{III}$.



- 2 Use the diagram to write an algebraic expression for the volume of each rectangular prism.

$$V_I = a^2(a - b)$$

$$V_{II} = ab(a - b)$$

$$V_{III} = b^2(a - b)$$

- 3 Write the equation for $a^3 - b^3$ by using the expressions for the rectangular prisms labeled I, II, and III from Problem 2. Factor to get the factored form of $a^3 - b^3$.

$$a^3 - b^3 = V_I + V_{II} + V_{III}$$

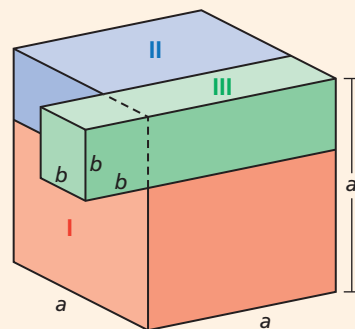
$$a^3 - b^3 = a^2(a - b) + ab(a - b) + b^2(a - b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Try This

The figure shown is a large cube with a small cube added to one corner.

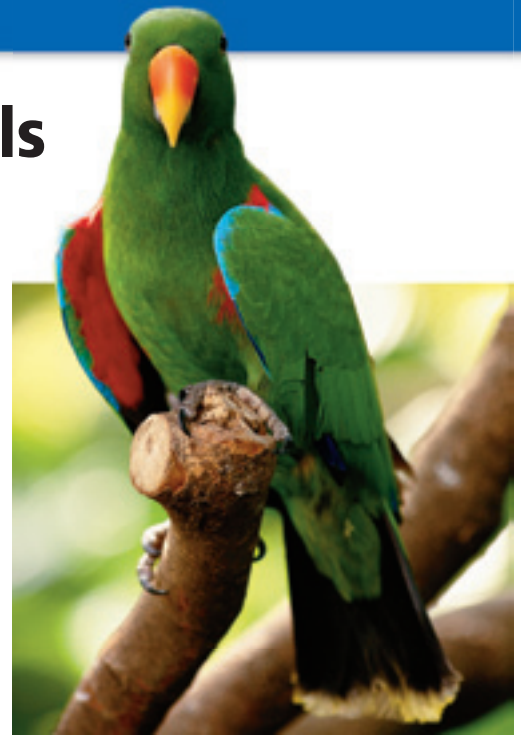
1. Explain why the volume of the figure is $a^3 + b^3$. How is this related to the volumes of the rectangular prisms labeled I, II, and III?
2. Use the diagram to write an algebraic expression for the volume of each rectangular prism.
3. Write the equation for $a^3 + b^3$ by using the expressions for the rectangular prisms labeled I, II, and III from Problem 2. Factor to get the factored form of $a^3 + b^3$. (Hint: Factor the expressions for I and II before adding the expression for III.)





6-4

Factoring Polynomials



Objectives

Use the Factor Theorem to determine factors of a polynomial.

Factor the sum and difference of two cubes.

Who uses this?

Ecologists may use factoring polynomials to determine when species might become extinct. (See Example 4.)

Recall that if a number is divided by any of its factors, the remainder is 0. Likewise, if a polynomial is divided by any of its factors, the remainder is 0.

The Remainder Theorem states that if a polynomial is divided by $(x - a)$, the remainder is the value of the function at a . So, if $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.



Factor Theorem

THEOREM	EXAMPLE
For any polynomial $P(x)$, $(x - a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.	Because $P(1) = 1^2 - 1 = 0$, $(x - 1)$ is a factor of $P(x) = x^2 - 1$.

EXAMPLE 1

Determining Whether a Linear Binomial is a Factor

Determine whether the given binomial is a factor of the polynomial $P(x)$.

A $(x - 3); P(x) = x^2 + 2x - 3$

Find $P(3)$ by synthetic substitution.

$$\begin{array}{r|rrrr} 3 & 1 & 2 & -3 & \\ & & 3 & 15 & \\ \hline & 1 & 5 & 12 & \end{array}$$

$P(3) = 12$

$P(3) \neq 0$, so $(x - 3)$ is not a factor of $P(x) = x^2 + 2x - 3$.

B $(x + 4); P(x) = 2x^4 + 8x^3 + 2x + 8$

Find $P(-4)$ by synthetic substitution.

$$\begin{array}{r|rrrrrr} -4 & 2 & 8 & 0 & 2 & 8 \\ & & -8 & 0 & 0 & -8 \\ \hline & 2 & 0 & 0 & 2 & 0 \end{array}$$

$P(-4) = 0$, so $(x + 4)$ is a factor of $P(x) = 2x^4 + 8x^3 + 2x + 8$.



Determine whether the given binomial is a factor of the polynomial $P(x)$.

1a. $(x + 2); P(x) = 4x^2 - 2x + 5$

1b. $(3x - 6); P(x) = 3x^4 - 6x^3 + 6x^2 + 3x - 30$

You are already familiar with methods for factoring quadratic expressions. You can factor polynomials of higher degrees using many of the same methods you learned in Lesson 5-3.

EXAMPLE 2 Factoring by GroupingFactor $x^3 + 3x^2 - 4x - 12$.

$$(x^3 + 3x^2) + (-4x - 12)$$

Group terms.

$$x^2(x + 3) - 4(x + 3)$$

Factor common monomials from each group.

$$(x + 3)(x^2 - 4)$$

Factor out the common binomial $(x + 3)$.

$$(x + 3)(x + 2)(x - 2)$$

*Factor the difference of squares.***Check** Use the table feature of your calculator to compare the original expression and the factored form.

The table shows that the original function and the factored form have the same function values. ✓



Factor each expression.

2a. $x^3 - 2x^2 - 9x + 18$

2b. $2x^3 + x^2 + 8x + 4$

Just as there is a special rule for factoring the difference of two squares, there are special rules for factoring the sum or difference of two cubes.

**Factoring the Sum and the Difference of Two Cubes**

METHOD	ALGEBRA
Sum of two cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of two cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

EXAMPLE 3 Factoring the Sum or Difference of Two Cubes

Factor each expression.

A $5x^4 + 40x$

$$5x(x^3 + 8)$$

Factor out the GCF, $5x$.

$$5x(x^3 + 2^3)$$

Rewrite as the sum of cubes.

$$5x(x + 2)(x^2 - x \cdot 2 + 2^2)$$

Use the rule $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$5x(x + 2)(x^2 - 2x + 4)$$

B $8y^3 - 27$

$$(2y)^3 - 3^3$$

Rewrite as the difference of cubes.

$$(2y - 3)[(2y)^2 + 2y \cdot 3 + 3^2]$$

Use the rule $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$$(2y - 3)(4y^2 + 6y + 9)$$

Remember!

GCF stands for "greatest common factor." Always factor out the GCF before using other methods.



Factor each expression.

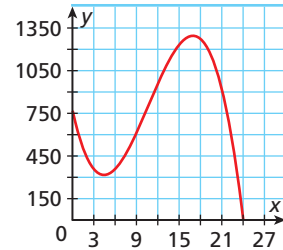
3a. $8 + z^6$

3b. $2x^5 - 16x^2$

You can also use a graph to help you factor a polynomial. Recall that the real zeros of a function appear as x -intercepts on its graph. By the Factor Theorem, if you can determine the zeros of a polynomial function from its graph, you can determine the corresponding factors of the polynomial.

EXAMPLE 4 Ecology Application

The population of an endangered species of bird in the years since 1990 can be modeled by the function $P(x) = -x^3 + 32x^2 - 224x + 768$. Identify the year that the bird will become extinct if the model is accurate and no protective measures are taken. Use the graph to factor $P(x)$.



Because $P(x)$ represents the population, the real zero of $P(x)$ represents a population of zero, meaning extinction. $P(x)$ has only one real zero at $x = 24$, which corresponds to the year 2014.

If the model is accurate, the bird will become extinct in 2014.

The corresponding factor is $(x - 24)$.

$$\begin{array}{r|rrrrr} 24 & -1 & 32 & -224 & 768 & \\ & & -24 & 192 & -768 & \\ \hline & -1 & 8 & -32 & 0 & \end{array}$$

Use synthetic division to factor the polynomial.

$$P(x) = (x - 24)(-x^2 + 8x - 32)$$

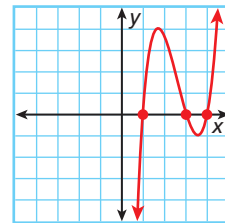
Write $P(x)$ as a product.

$$P(x) = -(x - 24)(x^2 - 8x + 32)$$

Factor out -1 from the quadratic.



4. The volume of a rectangular prism is modeled by the function $V(x) = x^3 - 8x^2 + 19x - 12$, which is graphed at right. Identify the values of x for which $V(x) = 0$, and use the graph to factor $V(x)$.



THINK AND DISCUSS

1. Explain how to use the Factor Theorem to determine whether a linear binomial is a factor of a polynomial.
2. Explain how you know when to use the sum or difference of cubes to factor a binomial.
3. **GET ORGANIZED** Copy and complete the graphic organizer. For each method, give an example of a polynomial and its factored form.



Method	Polynomial	Factored Form
Difference of Two Squares		
Difference of Two Cubes		
Sum of Two Cubes		

GUIDED PRACTICE

SEE EXAMPLE 1

p. 430

Determine whether the given binomial is a factor of the polynomial $P(x)$.

- $(x + 1); P(x) = 2x^4 + 2x^3 - x^2 - 5x - 4$
- $(x - 2); P(x) = 5x^3 + x^2 - 7$
- $(2x - 4); P(x) = 2x^5 - 4x^4 + 2x^2 - 2x - 4$

SEE EXAMPLE 2

p. 431

Factor each expression.

- $x^3 + x^2 - x - 1$
- $x^3 + 5x^2 - 4x - 20$
- $8x^3 + 4x^2 - 2x - 1$
- $2x^3 - 2x^2 - 8x + 8$
- $2x^3 - 3x^2 - 2x + 3$
- $12x^2 + 3x - 24x - 6$
- $8 - m^6$
- $2t^7 + 54t^4$
- $x^3 + 64$
- $27 + x^3$
- $4t^5 - 32t^2$
- $y^3 - 125$

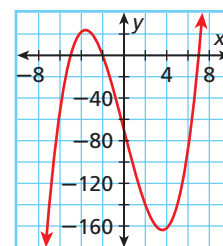
SEE EXAMPLE 3

p. 431

SEE EXAMPLE 4

p. 432

16. The volume of a cargo container is modeled by the function $V(x) = x^3 - 39x - 70$. Identify the values of x for which $V(x) = 0$, and use the graph to factor $V(x)$.



PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
17–19	1
20–25	2
26–31	3
32	4

Determine whether the given binomial is a factor of the polynomial $P(x)$.

- $(x - 3); P(x) = 4x^6 - 12x^5 + 2x^3 - 6x^2 - 5x + 10$
- $(x - 8); P(x) = x^5 - 8x^4 + 8x - 64$
- $(3x + 12); P(x) = 3x^4 + 12x^3 + 6x + 24$

Extra Practice

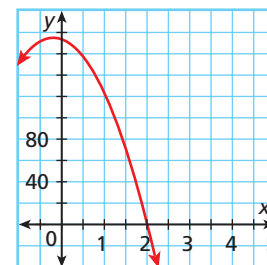
Skills Practice p. 514

Application Practice p. 537

Factor each expression.

- $8y^3 - 4y^2 - 50y + 25$
- $4b^3 + 3b^2 - 16b - 12$
- $3p^3 - 21p^2 - p + 7$
- $3x^3 + x^2 - 27x - 9$
- $8z^2 - 4z + 10z - 5$
- $5x^3 - x^2 - 20x + 4$
- $125 + z^3$
- $s^6 - 1$
- $24n^2 + 3n^5$
- $6x^4 - 162x$
- $40 - 5t^3$
- $y^5 + 27y^2$

32. **Recreation** The volume of a bowling ball can be modeled by the function $V(x) = 168 - 28x - 28x^2$, where x represents the radius of the finger holes in inches. Identify the values of x for which $V(x) = 0$, and use the graph to factor $V(x)$.



Factor completely.

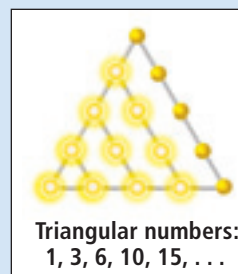
- $x^6 - 14x^4 + 49x^2$
- $2x^3 + x^2 - 72x - 36$
- $4x^3 + x^2 - 16x - 4$
- $9x^9 - 16x^7 + 9x^6 - 16x^4$
- $8x^7 - 4x^5 - 18x^3 + 9x$
- $x^{13} - 15x^9 - 16x^5$
- Critical Thinking** The polynomial $ax^3 + bx^2 + cx + d$ is factored as $3(x - 2)(x + 3)(x - 4)$. What are the values of a and d ? Explain.



40. This problem will prepare you for the Multi-Step Test Prep on page 436.

The total number of lights in a triangular lighting rig is related to the triangular numbers, as shown at right. The sum of the first n triangular numbers is given by the polynomial function $g(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$.

- Find the sum of the first seven triangular numbers.
- Explain why $n - 7$ must be a factor of $n^3 + 3n^2 + 2n - 504$.
- Factor $n^3 + 3n^2 + 2n - 504$.



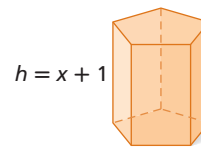
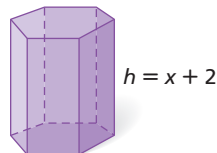
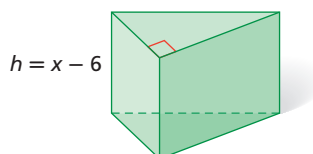
Use the Factor Theorem to verify that each linear binomial is a factor of the given polynomial. Then use synthetic division to write the polynomial as a product.

- $(x - 2); P(x) = x^4 - 2x^3 + 5x^2 - 9x - 2$.
- $(x - 1); P(x) = 4x^6 - 4x^5 - 2x^4 + 3x^3 - x^2 - 7x + 7$.
- $(x + 2); P(x) = 2x^5 + 4x^4 - 6x^2 - 9x + 6$.
- $(x - 4); P(x) = 2x^4 - 9x^3 + 7x^2 - 14x + 8$.
- Business** The profit of a small business (in thousands of dollars) since it was founded can be modeled by the polynomial $f(t) = -t^4 + 44t^3 - 612t^2 + 2592t$, where t represents the number of years since 1980.
 - Factor $f(t)$ completely.
 - What was the company's profit in 1985?
 - Find and interpret $f(15)$.
 - What can you say about the company's long-term prospects?



Geometry Given the volume and height, find a polynomial expression for the area of the base in terms of x for each figure. (Hint: $V = Bh$)

46. $V(x) = 2x^3 - 17x^2 + 27x + 18$ 47. $V(x) = x^4 - 16$ 48. $V(x) = 3x^6 + 3x^3$



49. **Write About It** Describe how synthetic division can be used to factor a polynomial.



- Which is a factor of $x^3 + 2x^2 - 9x + 30$?

(A) $x + 2$ (B) $x - 3$ (C) $x + 5$ (D) $x - 6$
- $P(x)$ is a polynomial, and $P(4) = P(-2) = P(-1) = 0$. Which of the following could be $P(x)$?

(F) $x^3 + 7x^2 + 14x + 8$ (H) $x^2 + 3x + 2$
 (G) $-x^2 + 2x + 8$ (J) $x^3 - x^2 - 10x - 8$
- Short Response** Factor $4p^5 - 16p^3 - 20p$ completely.

- The polynomial $au^2 + bu + c$ is in quadratic form when u is any function of x . Identify u , and factor each expression, simplifying the factors if possible.

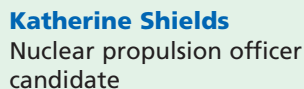
59. $\frac{1}{2}\left(x - \frac{1}{3}\right)^2 + \frac{5}{2}\left(x - \frac{1}{3}\right) - 42$

- Simplify.** Write the result in the form $a + bi$. (Lesson 5-9)

64. $\frac{8 - 2i}{i}$

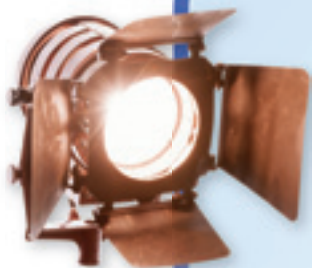
68. $P(x) = 6x^2 - 3x - 8$ for $x = 3$

go.hrw.com
Career Resources Online
KEYWORD: MB7 Career



A: There are many options available after my nuclear officer training. While serving as an officer, I'd like to go to graduate school for an advanced degree in nuclear engineering.

MULTI-STEP TEST PREP



Operations with Polynomials

In the Spotlight A lighting rig is a large bank of lights that is used to create lighting effects at concerts and sporting events. A company makes rigs with lights arranged in polygonal patterns. The number of lights in a rig depends on the shape of the rig and the number of rows of lights. For example, a triangular rig may have 1, 3, 6, 10, or 15 lights, depending on the number of rows. The figures show the number of lights in a variety of rigs.



Triangular rig
Number of lights:
1, 3, 6, 10, 15, ...



Square rig
Number of lights:
1, 4, 9, 16, 25, ...



Pentagonal rig
Number of lights:
1, 5, 12, 22, 35, ...



Hexagonal rig
Number of lights:
1, 6, 15, 28, 45, ...

- The number of lights in a triangular rig with n rows is given by $T(n) = \frac{1}{2}n^2 + \frac{1}{2}n$. Find the number of lights in triangular rigs with 1 to 10 rows.
- The number of lights in a square rig with n rows is given by $S(n) = n^2$. Find the number of lights in square rigs with 1 to 10 rows.
- The number of lights in a pentagonal rig with n rows is given by $P(n) = \frac{3}{2}n^2 - \frac{1}{2}n$. Find the number of lights in pentagonal rigs with 1 to 10 rows.
- Write a polynomial function, $H(n)$, that gives the number of lights in a hexagonal rig with n rows. (*Hint:* Look for a pattern in the coefficients of the polynomial functions in parts a–c.)
- Find $T(n) + P(n)$. What do you notice about this sum?
- Find $H(n) \div S(n)$. What can you say about the ratio of the number of lights in a hexagonal rig with n rows to the number of lights in a square rig with n rows as n gets larger and larger?



Quiz for Lesson 6-1 Through 6-4



6-1 Polynomials

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

1. $4x^2 + 3x^5 - 5$ 2. $7 + 13x$ 3. $1 + 5x^3 + x^2 - 3x$ 4. $8x + 2x^4 - 5x^3$

Add or subtract. Write your answer in standard form.

5. $(3x^2 + 1) + (4x^2 + 3)$ 6. $(9x^3 - 6x^2) - (2x^3 + x^2 + 2)$
 7. $(11x^2 + x^3 + 7) + (5x^3 + 4x^2 - 2x)$ 8. $(x^5 - 4x^4 + 1) - (-7x^4 + 11)$
 9. The cost of manufacturing x units of a product can be modeled by $C(x) = x^3 - 15x + 15$. Evaluate $C(x)$ for $x = 100$, and describe what the value represents.

Graph each polynomial function on a calculator. Describe the graph, and identify the number of real zeros.

10. $f(x) = -\frac{1}{4}x^5 - x^2$ 11. $h(x) = \frac{1}{5}x^3 + x^2 - 2$ 12. $f(x) = -2x^6 - 1$



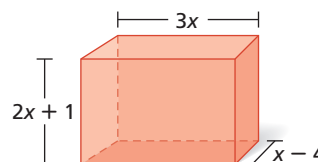
6-2 Multiplying Polynomials

Find each product.

13. $2y(4x^2 + 7xy)$ 14. $(a + b)(3ab + b^2)$
 15. $(2x + \frac{1}{3})^2$ 16. $(2x - 3)(x^3 - x^2 + 3x + 5)$

Expand each expression.

17. $(x - 3)^4$ 18. $(x + 2y)^3$ 19. $(4x - 1)^4$
 20. Find a polynomial expression in terms of x for the volume of the rectangular prism shown.



6-3 Dividing Polynomials

Divide.

21. $(6y^2 + 13y - 8) \div (2y - 1)$ 22. $(3x^3 + 11x^2 + 11x + 15) \div (x + 3)$

Use synthetic substitution to evaluate the polynomial for the given value.

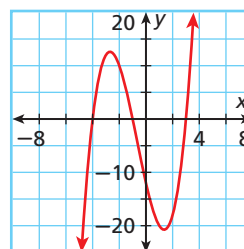
23. $P(x) = x^3 + 2x^2 - 5x + 6$ for $x = -1$ 24. $P(x) = x^4 + x^2 + x - 6$ for $x = 2$



6-4 Factoring Polynomials

Factor each expression.

25. $3t^3 - 21t^2 - 12t$ 26. $16y^2 - 49$
 27. $y^3 + 7y^2 + 2y + 14$ 28. $a^6 + 125$
 29. The volume of a box is modeled by the function $V(x) = x^3 + 2x^2 - 11x - 12$. Identify the values of x for which the volume is 0 and use the graph to factor $V(x)$.





6-5

Finding Real Roots of Polynomial Equations

Objectives

Identify the multiplicity of roots.

Use the Rational Root Theorem and the Irrational Root Theorem to solve polynomial equations.

Vocabulary

multiplicity

Who uses this?

Package designers can use roots of polynomial equations to set production specifications. (See Example 3.)

In Lesson 6-4, you used several methods for factoring polynomials. As with some quadratic equations, factoring a polynomial equation is one way to find its real roots.

Recall the Zero Product Property from Lesson 5-3. You can find the *roots*, or *solutions*, of the polynomial equation $P(x) = 0$ by setting each factor equal to 0 and solving for x .



EXAMPLE 1 Using Factoring to Solve Polynomial Equations

Solve each polynomial equation by factoring.

A $3x^5 + 18x^4 + 27x^3 = 0$

$$3x^3(x^2 + 6x + 9) = 0$$

Factor out the GCF, $3x^3$.

$$3x^3(x + 3)(x + 3) = 0$$

Factor the quadratic.

$$3x^3 = 0, x + 3 = 0, \text{ or } x + 3 = 0$$

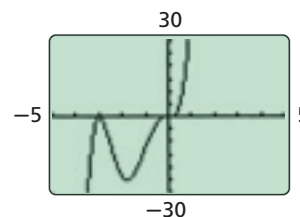
Set each factor equal to 0.

$$x = 0, x = -3, \text{ or } x = -3$$

Solve for x .

The roots are 0 and -3 .

Check Use a graph. The roots appear to be located at $x = 0$ and $x = -3$. ✓



B $x^4 - 13x^2 = -36$

$$x^4 - 13x^2 + 36 = 0$$

Set the equation equal to 0.

$$(x^2 - 4)(x^2 - 9) = 0$$

Factor the trinomial in quadratic form.

$$(x + 2)(x - 2)(x + 3)(x - 3) = 0$$

Factor the difference of two squares.

$$x + 2 = 0, x - 2 = 0, x + 3 = 0, \text{ or } x - 3 = 0$$

$$x = -2, x = 2, x = -3, \text{ or } x = 3$$

The roots are $-2, 2, -3$, and 3 .



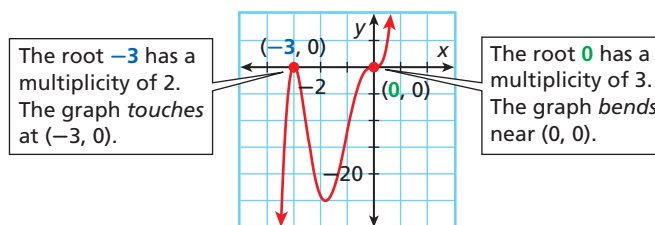
Solve each polynomial equation by factoring.

1a. $2x^6 - 10x^5 - 12x^4 = 0$

1b. $x^3 - 2x^2 - 25x = -50$

Sometimes a polynomial equation has a factor that appears more than once. This creates a *multiple root*. In Example 1A, $3x^5 + 18x^4 + 27x^3 = 0$ has two multiple roots, 0 and -3 . For example, the root 0 is a factor **three** times because $3x^3 = 0$.

The **multiplicity** of root r is the number of times that $x - r$ is a factor of $P(x)$. When a real root has even multiplicity, the graph of $y = P(x)$ touches the x -axis but does not cross it. When a real root has odd multiplicity greater than 1, the graph “bends” as it crosses the x -axis.



You cannot always determine the multiplicity of a root from a graph. It is easiest to determine multiplicity when the polynomial is in factored form.

EXAMPLE 2 Identifying Multiplicity

Identify the roots of each equation. State the multiplicity of each root.

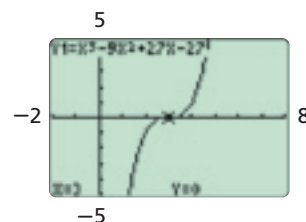
A $x^3 - 9x^2 + 27x - 27 = 0$

$$x^3 - 9x^2 + 27x - 27 = (x - 3)(x - 3)(x - 3)$$

$x - 3$ is a factor **three** times.

The root **3** has a **multiplicity of 3**.

Check Use a graph. A calculator graph shows a bend near $(3, 0)$. ✓



B $-2x^3 - 12x^2 + 30x + 200 = 0$

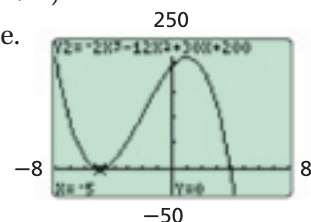
$$-2x^3 - 12x^2 + 30x + 200 = -2(x - 4)(x + 5)(x + 5)$$

$x - 4$ is a factor once, and $x + 5$ is a factor twice.

The root 4 has a multiplicity of 1.

The root -5 has a multiplicity of 2.

Check Use a graph. The graph crosses at $(4, 0)$ and touches at $(-5, 0)$. ✓



Identify the roots of each equation. State the multiplicity of each root.

2a. $x^4 - 8x^3 + 24x^2 - 32x + 16 = 0$

2b. $2x^6 - 22x^5 + 48x^4 + 72x^3 = 0$

Not all polynomials are factorable, but the Rational Root Theorem can help you find all possible rational roots of a polynomial equation.



Rational Root Theorem

If the polynomial $P(x)$ has integer coefficients, then every rational root of the polynomial equation $P(x) = 0$ can be written in the form $\frac{p}{q}$, where p is a factor of the constant term of $P(x)$ and q is a factor of the leading coefficient of $P(x)$.

EXAMPLE 3 Marketing Application

A popcorn producer is designing a new box for the popcorn. The marketing department has designed a box with the width 2 inches less than the length and with the height 5 inches greater than the length. The volume of each box must be 24 cubic inches. What is the length of the box?



Step 1 Write an equation to model the volume of the box.

Let x represent the length in inches. Then the width is $x - 2$, and the height is $x + 5$.

$$x(x - 2)(x + 5) = 24 \quad V = \ell wh.$$

$$x^3 + 3x^2 - 10x = 24 \quad \text{Multiply the left side.}$$

$$x^3 + 3x^2 - 10x - 24 = 0 \quad \text{Set the equation equal to 0.}$$

Step 2 Use the Rational Root Theorem to identify all possible rational roots.

Factors of -24 : $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

Step 3 Test the possible roots to find one that is actually a root. The length must be positive, so try only positive rational roots.

$\frac{p}{q}$	1	3	-10	-24
1	1	4	-6	-30
2	1	5	0	-24
3	1	6	8	0
4	1	7	18	48

Use a synthetic substitution table to organize your work. The first row represents the coefficients of the polynomial. The first column represents the divisors and the last column represents the remainders. Test divisors to identify at least one root.

Step 4 Factor the polynomial. The synthetic substitution of 3 results in a remainder of 0, so 3 is a root and the polynomial in factored form is $(x - 3)(x^2 + 6x + 8)$.

$$(x - 3)(x^2 + 6x + 8) = 0 \quad \text{Set the equation equal to 0.}$$

$$(x - 3)(x + 2)(x + 4) = 0 \quad \text{Factor } x^2 + 6x + 8.$$

$$x = 3, x = -2, \text{ or } x = -4 \quad \text{Set each factor equal to 0, and solve.}$$

The length must be positive, so the length should be 3 inches.

Check Substitute 3 for x in the formula for volume.

$$x(x - 2)(x + 5) = 24$$

$$3(3 - 2)(3 + 5) = 24$$

$$24 = 24 \quad \checkmark$$

Helpful Hint

In Example 3, substitute 3 for x to check your (answer).

$$\begin{array}{r|l} x(x - 2)(x + 5) = 24 & \\ 3(3 - 2)(3 + 5) & 24 \\ 3(1)(8) & 24 \\ 24 & 24 \checkmark \end{array}$$



3. A shipping crate must hold 12 cubic feet. The width should be 1 foot less than the length, and the height should be 4 feet greater than the length. What should the length of the crate be?

Polynomial equations may also have irrational roots.



Irrational Root Theorem

If the polynomial $P(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the polynomial equation $P(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $P(x) = 0$.

The Irrational Root Theorem says that irrational roots of the form $a + b\sqrt{c}$ come in conjugate pairs. For example, if you know that $1 + \sqrt{2}$ is a root of $x^3 - x^2 - 3x - 1 = 0$, then you know that $1 - \sqrt{2}$ is also a root.

Recall that the real numbers are made up of the rational and the irrational numbers. You can use the Rational Root Theorem and the Irrational Root Theorem together to find *all* of the real roots of $P(x) = 0$.

EXAMPLE 4 Identifying All of the Real Roots of a Polynomial Equation

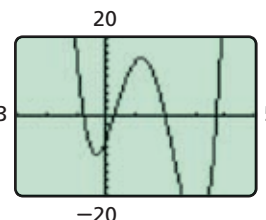
Identify all of the real roots of $4x^4 - 21x^3 + 18x^2 + 19x - 6 = 0$.

Step 1 Use the Rational Root Theorem to identify possible rational roots.

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4} \quad \begin{array}{l} p = -6 \text{ and} \\ q = 4 \end{array}$$

Step 2 Graph $y = 4x^4 - 21x^3 + 18x^2 + 19x - 6$ to find the x -intercepts.

The x -intercepts are located at or near -0.75 , 0.27 , 2 , and 3.73 . The x -intercepts 0.27 and 3.73 do not correspond to any of the possible rational roots.



Step 3 Test the possible rational roots 2 and $-\frac{3}{4}$.

$$\begin{array}{r|rrrrr} 2 & 4 & -21 & 18 & 19 & -6 \\ & & 8 & -26 & -16 & 6 \\ \hline & 4 & -13 & -8 & 3 & 0 \end{array}$$

Test 2. The remainder is 0, so $(x - 2)$ is a factor.

The polynomial factors into $(x - 2)(4x^3 - 13x^2 - 8x + 3)$.

$$\begin{array}{r|rrrr} -\frac{3}{4} & 4 & -13 & -8 & 3 \\ & & -3 & 12 & -3 \\ \hline & 4 & -16 & 4 & 0 \end{array}$$

Test $-\frac{3}{4}$ in the cubic polynomial. The remainder is 0, so $(x + \frac{3}{4})$ is a factor.

The polynomial factors into $(x - 2)(x + \frac{3}{4})(4x^2 - 16x + 4)$.

Step 4 Solve $4x^2 - 16x + 4 = 0$ to find the remaining roots.

$$4(x^2 - 4x + 1) = 0$$

Factor out the GCF of 4.

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

Use the quadratic formula to identify the irrational roots.

The fully factored equation is $4(x - 2)(x + \frac{3}{4})[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0$.

The roots are 2 , $-\frac{3}{4}$, $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

Helpful Hint

In Example 4, the x -intercepts 0.27 and 3.73 correspond to the irrational roots $2 - \sqrt{3}$ and $2 + \sqrt{3}$.



4. Identify all of the real roots of $2x^3 - 3x^2 - 10x - 4 = 0$.

THINK AND DISCUSS

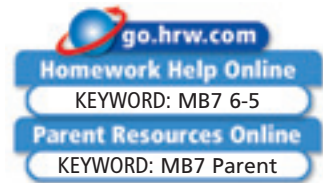


1. Explain how to recognize the multiplicity of a root of a polynomial in factored form.
2. **GET ORGANIZED** Copy and complete the graphic organizer. Give roots that satisfy each theorem and write a polynomial equation that has those roots.

Theorem	Roots	Polynomial
Rational Root Theorem		
Irrational Root Theorem		

6-5

Exercises



GUIDED PRACTICE

1. **Vocabulary** Explain how *multiplicity* is related to the word *multiple*.

SEE EXAMPLE 1

p. 438

1. Solve each polynomial equation by factoring.

$$2. 2x^4 + 16x^3 + 32x^2 = 0$$

$$3. x^4 - 37x^2 + 36 = 0$$

$$4. 4x^7 - 28x^6 = -48x^5$$

$$5. 3x^4 + 11x^3 = 4x^2$$

$$6. 2x^3 - 12x^2 = 32x - 192$$

$$7. x^4 + 100 = 29x^2$$

SEE EXAMPLE 2

p. 439

2. Identify the roots of each equation. State the multiplicity of each root.

$$8. 2x^5 + 12x^4 + 16x^3 - 12x^2 - 18x = 0$$

$$9. x^6 - 12x^4 + 48x^2 - 64 = 0$$

SEE EXAMPLE 3

p. 440

10. **Storage** A cedar chest has a length that is 3 feet longer than its width and a height that is 1 foot longer than its width. The volume of the chest is 30 cubic feet. What is the width?

SEE EXAMPLE 4

p. 440

- Identify all of the real roots of each equation.

$$11. x^3 + 6x^2 - 5x - 30 = 0$$

$$12. 3x^3 - 18x^2 - 9x + 132 = 0$$

$$13. 2x^3 - 42x + 40 = 0$$

$$14. x^4 - 9x^2 + 20 = 0$$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises See Example

15–20 1

21–22 2

23 3

24–26 4

Extra Practice

Skills Practice p. S14

Application Practice p. S37

- Solve each polynomial equation by factoring.

$$15. x^3 + 3x^2 - 9x = 27$$

$$16. 4x^5 - 8x^3 + 4x = 0$$

$$17. 10x^3 - 640x = 0$$

$$18. x^4 - 12x^2 = -36$$

$$19. 2x^3 - 5x^2 - 4x + 10 = 0$$

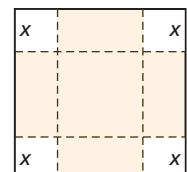
$$20. 4x^3 + 7x^2 - 5x = 6$$

- Identify the roots of each equation. State the multiplicity of each root.

$$21. 8x^5 - 192x^4 + 1536x^3 - 4096x^2 = 0$$

$$22. x^4 + 2x^3 - 11x^2 - 12x + 36 = 0$$

23. **Measurement** An open box is to be made from a square piece of material with a side length of 10 inches by cutting equal squares from the corners and turning up the sides. What size of square would you cut out if the volume of the box must be 48 cubic inches?



Identify all of the real roots of each equation.

24. $x^4 - 3x^2 - 4 = 0$

25. $3x^3 + 4x^2 - 6x - 8 = 0$

26. $x^4 - 2x^3 - 2x^2 = 0$



27. **Graphing Calculator** Consider the polynomial function

$$f(x) = x^4 + 3x^3 - 3x^2 - 12x - 4.$$

- Use the Rational Root Theorem to list the possible rational roots of this equation.
- Graph the polynomial on a graphing calculator. Which possible rational roots are zeros of $f(x)$?
- According to the graph, how many other real zeros does the function have?
- Approximate these zeros to the nearest hundredth by using the zero feature.

28. **Multi-Step** A manufacturing company must design a box that holds 55 cubic inches. The width must be 4 inches less than the length, and the height must be 10 inches greater than the width.

- Let the length equal $x + 1$, and write expressions in x for the width and height of the box.
- Write a polynomial equation to represent the volume of the box.
- Identify all possible rational roots of the equation $V(x) = 55$.
- Factor the polynomial. What are the roots of the equation?
- What are the dimensions of the box?



Entertainment



The Sheikra roller coaster at Busch Gardens in Tampa is Florida's tallest roller coaster and includes a 138 ft dive into an underground tunnel.

Identify all of the real roots of each equation.

29. $x^3 - 7x^2 + 14x - 6 = 0$

30. $\frac{5}{3}x^3 + \frac{8}{3}x^2 - \frac{4}{3}x = 0$

31. $x^4 - x^3 - 31x^2 + 25x + 150 = 0$

32. $3x^4 + 19x^2 + 27x + 6 = 23x^3$

33. $x^5 - 4x^4 - 2x^3 + 4x^2 + x = 0$

34. $x^3 + 9 - 6x^2 = -4(11x - 2x^2)$

35. **Entertainment** The paths of some roller coasters may be modeled by a polynomial function, where t is the time, in tens of seconds, after the ride has started and $h(t)$ is the height, in feet. Some roller coasters go underground as well as above the ground. In factored form, the beginning part of one roller-coaster ride can be modeled by the function $h(t) = \frac{1}{4}(t - 2)(t - 4)(t - 7)(t - 9)$.

- What is the starting height of this roller coaster?
- Graph this function on your graphing calculator, and describe the path of the roller coaster for the first 100 seconds.
- Write an equation of a polynomial function that can be used to model a portion of a roller-coaster ride when the coaster starts 45 feet above the ground, enters an underground tunnel after 30 seconds, and then emerges from underground 20 seconds later.

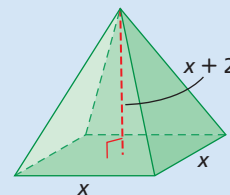
MULTI-STEP TEST PREP



36. This problem will prepare you for the Multi-Step Test Prep on page 472.

A type of cheese is packaged in a cardboard box shaped like a pyramid. As shown in the figure, the height is 2 cm greater than the length of the base.

- Write a polynomial function for the volume of the box.
- The volume of the box must be 147 cm^3 . Write a polynomial equation with integer coefficients that you can solve in order to find the length of the base.
- What are the dimensions of the box?



37. **Critical Thinking** Suppose you are looking at a graph to identify which possible rational roots correspond to zeros. You see an x -intercept near 4. You have 4 as a possible rational root, but $P(4) \neq 0$. What can you say about the zero that you located on the graph?



38. **Write About It** Graph the functions $f(x) = (x - 2)^2(x + 2)^3$, $g(x) = (x - 2)^2$, and $h(x) = (x + 2)^3$ on your graphing calculator. How does the behavior of f compare to the behavior of g near $x = 2$? How does the behavior of f compare to the behavior of h near $x = -2$? Explain.



39. **Write About It** How does the graph of $5x^4 - 20x^3$ behave near $(0, 0)$? How can you determine the behavior by factoring the expression?



40. Find the solutions to the equation $8x^3 - 2x^2 - 43x + 30 = 0$.

(A) 2 and $\frac{-7 \pm \sqrt{42}}{8}$

(C) 2, 15, and 1

(B) 2, $-\frac{5}{2}$, and $\frac{3}{4}$

(D) 1, 2, and -3

41. How many real zeros does the polynomial $f(x) = 4x^2 - 3x + 2x^4 - 5x^3 - 7$ have?

(F) 2

(G) 3

(H) 4

(J) 5

42. Which of the following is NOT a factor of $g(x) = 6x^3 + 13x^2 - 4$?

(A) $x + 2$

(B) $2x - 1$

(C) $x - 3$

(D) $3x + 2$

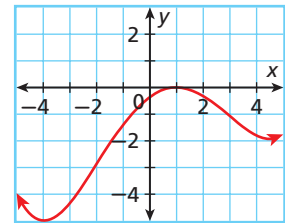
43. Use the graph shown at right to identify the multiplicity of the roots of $f(x) = 0$.

(F) Root 2 with a multiplicity of 1

(G) Root 2 with a multiplicity of 2

(H) Root 1 with a multiplicity of 2

(J) Root 1 with a multiplicity of 3



CHALLENGE AND EXTEND

44. Give a polynomial function that has the zeros 0, 1, and $3 - \sqrt{5}$.

Identify the value of k that makes the x -value a solution to the cubic equation.

45. $x^3 + 3x^2 - x + k = 0$; $x = 2$

46. $kx^3 - 2x^2 + x - 6 = 0$; $x = -3$

47. $6x^3 - 23x^2 - kx + 8 = 0$; $x = 4$

SPIRAL REVIEW

48. **Physics** A water balloon is dropped from the top of an 85-foot-tall building. The height of the water balloon in feet after t seconds is modeled by $h(t) = 85 - 16t^2$. What is the height of the water balloon after 2 seconds? (Lesson 5-3)

Solve each inequality by using algebra. (Lesson 5-7)

49. $x^2 + x + 5 > 11$

50. $x^2 - 10x + 2 \leq -23$

51. $x^2 < 1$

Factor each expression. (Lesson 6-4)

52. $x^3 + 3x^2 - 4x - 12$

53. $8x^3 + 4x^2 - 8x - 4$

54. $x^3 + 27$



6-6

Fundamental Theorem of Algebra

Objectives

Use the Fundamental Theorem of Algebra and its corollary to write a polynomial equation of least degree with given roots.

Identify all of the roots of a polynomial equation.

Who uses this?

Aerospace engineers may find roots of polynomial equations to determine dimensions of rockets. (See Example 4.)

You have learned several important properties about real roots of polynomial equations.



The following statements are equivalent:

A real number r is a root of the polynomial equation $P(x) = 0$.

$$P(r) = 0$$

r is an x -intercept of the graph of $P(x)$.

$x - r$ is a factor of $P(x)$.

When you divide the rule for $P(x)$ by $x - r$, the remainder is 0.

r is a zero of $P(x)$.

You can use this information to write a polynomial function when given its zeros.

EXAMPLE 1 Writing Polynomial Functions Given Zeros

Write the simplest polynomial function with zeros -3 , $\frac{1}{2}$, and 1 .

$$P(x) = (x + 3)\left(x - \frac{1}{2}\right)(x - 1) \quad \text{If } r \text{ is a zero of } P(x), \text{ then } x - r \text{ is a factor of } P(x).$$

$$P(x) = \left(x^2 + \frac{5}{2}x - \frac{3}{2}\right)(x - 1) \quad \text{Multiply the first two binomials.}$$

$$P(x) = x^3 + \frac{3}{2}x^2 - 4x + \frac{3}{2} \quad \text{Multiply the trinomial by the binomial.}$$

$$P(x) = x^3 + \frac{3}{2}x^2 - 4x + \frac{3}{2}$$



Write the simplest polynomial function with the given zeros.

1a. $-2, 2, 4$

1b. $0, \frac{2}{3}, 3$

Notice that the degree of the function in Example 1 is the same as the number of zeros. This is true for all polynomial functions. However, all of the zeros are not necessarily real zeros. Polynomial functions, like quadratic functions, may have complex zeros that are not real numbers.



The Fundamental Theorem of Algebra

Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.

Corollary: Every polynomial function of degree $n \geq 1$ has exactly n zeros, including multiplicities.

Using this theorem, you can write any polynomial function in factored form.

To find all roots of a polynomial equation, you can use a combination of the Rational Root Theorem, the Irrational Root Theorem, and methods for finding complex roots, such as the quadratic formula.

EXAMPLE 2 Finding All Roots of a Polynomial Equation

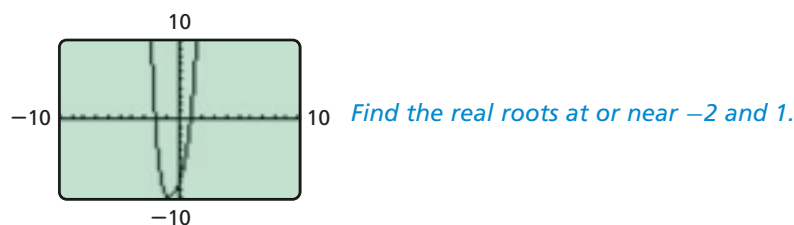
Solve $x^4 + x^3 + 2x^2 + 4x - 8 = 0$ by finding all roots.

The polynomial is of degree 4, so there are exactly four roots for the equation.

Step 1 Use the Rational Root Theorem to identify possible rational roots.

$$\frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1} = \pm 1, \pm 2, \pm 4, \pm 8 \quad p = -8 \text{ and } q = 1$$

Step 2 Graph $y = x^4 + x^3 + 2x^2 + 4x - 8$ to find the real roots.



Step 3 Test the possible real roots.

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & 2 & 4 & -8 \\ & & 1 & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array}$$

Test 1. The remainder is 0, so $(x - 1)$ is a factor.

The polynomial factors into $(x - 1)(x^3 + 2x^2 + 4x + 8) = 0$.

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Test -2 in the cubic polynomial. The remainder is 0, so $(x + 2)$ is a factor.

The polynomial factors into $(x - 1)(x + 2)(x^2 + 4) = 0$.

Step 4 Solve $x^2 + 4 = 0$ to find the remaining roots.

$$\begin{aligned} x^2 + 4 &= 0 \\ x^2 &= -4 \\ x &= \pm 2i \end{aligned}$$

The fully factored form of the equation is

$$(x - 1)(x + 2)(x + 2i)(x - 2i) = 0.$$

The solutions are 1 , -2 , $2i$, and $-2i$.



2. Solve $x^4 + 4x^3 - x^2 + 16x - 20 = 0$ by finding all roots.

The real numbers are a subset of the complex numbers, so a real number a can be thought of as the complex number $a + 0i$. But here the term *complex root* will only refer to a root of the form $a + bi$, where $b \neq 0$. Complex roots, like irrational roots, come in conjugate pairs. Recall from Chapter 5 that the complex conjugate of $a + bi$ is $a - bi$.



Complex Conjugate Root Theorem

If $a + bi$ is a root of a polynomial equation with real-number coefficients, then $a - bi$ is also a root.

EXAMPLE 3 Writing a Polynomial Function with Complex Zeros

Write the simplest polynomial function with zeros $1 + i$, $\sqrt{2}$, and -3 .

Step 1 Identify all roots.

By the Irrational Root Theorem and the Complex Conjugate Root Theorem, the irrational roots and complex roots come in conjugate pairs. There are five roots: $1 + i$, $1 - i$, $\sqrt{2}$, $-\sqrt{2}$, and -3 . The polynomial must have degree 5.

Step 2 Write the equation in factored form.

$$P(x) = [x - (1 + i)][x - (1 - i)](x - \sqrt{2})[x - (-\sqrt{2})][x - (-3)]$$

Step 3 Multiply.

$$P(x) = (x^2 - 2x + 2)(x^2 - 2)(x + 3)$$

$$= (x^4 - 2x^3 + 4x - 4)(x + 3)$$

$$P(x) = x^5 + x^4 - 6x^3 + 4x^2 + 8x - 12$$



3. Write the simplest polynomial function with zeros $2i$, $1 + \sqrt{2}$, and 3 .

EXAMPLE 4 Problem-Solving Application



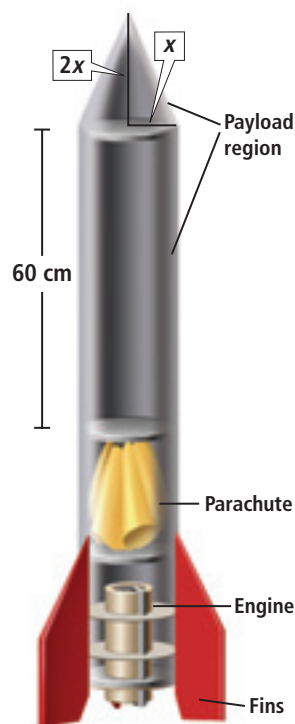
An engineering class is designing model rockets for a competition. The body of the rocket must be cylindrical with a cone-shaped top. The cylinder part must be 60 cm tall, and the height of the cone must be twice the radius. The volume of the payload region must be $558\pi \text{ cm}^3$ in order to hold the cargo. Find the radius of the rocket.

1 Understand the Problem

The cylinder and the cone have the same radius, x . The answer will be the value of x .

List the important information:

- The cylinder is 60 cm tall.
- The height of the cone part is twice the radius, $2x$.
- The volume of the payload region is $558\pi \text{ cm}^3$.



2 Make a Plan

Write an equation to represent the volume of the body of the rocket.

$$V = V_{\text{cone}} + V_{\text{cylinder}}$$

$$V(x) = \frac{2}{3}\pi x^3 + 60\pi x^2$$

$$V_{\text{cone}} = \frac{1}{3}\pi x^2 h \text{ and } V_{\text{cylinder}} = \pi x^2 h$$

Set the volume equal to 558π .

$$\frac{2}{3}\pi x^3 + 60\pi x^2 = 558\pi$$

3 Solve

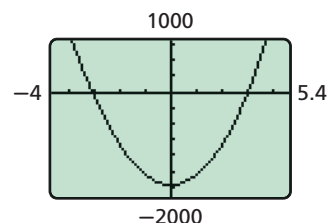
$$\frac{2}{3}\pi x^3 + 60\pi x^2 - 558\pi = 0$$

Write in
standard form.

$$\frac{2}{3}x^3 + 60x^2 - 558 = 0$$

Divide both
sides by π .

The graph indicates a possible positive root of 3. Use synthetic division to verify that 3 is a root, and write the equation as $(x - 3)\left(\frac{2}{3}x^2 + 62x + 186\right) = 0$. By the quadratic formula, you can find that -3.1 and -89.9 are approximate roots of $\frac{2}{3}x^2 + 62x + 186 = 0$. The radius must be a positive number, so the radius of the rocket is 3 cm.



$$\begin{array}{r|rrrr} 3 & \frac{2}{3} & 60 & 0 & -558 \\ & & 2 & 186 & 550 \\ \hline & \frac{2}{3} & 62 & 186 & 0 \end{array}$$

4 Look Back

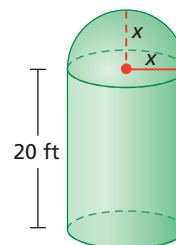
Substitute 3 cm into the original equation for the volume of the rocket.

$$V(3) = \frac{2}{3}\pi(3)^3 + 60\pi(3)^2$$

$$V(3) = 558\pi \quad \checkmark$$



4. A grain silo is in the shape of a cylinder with a hemisphere top. The cylinder is 20 feet tall. The volume of the silo is 2106π cubic feet. Find the radius of the silo.



THINK AND DISCUSS

- Explain why a polynomial equation with real coefficients and root $1 - i$ must be of degree two or greater.
- If $P(x)$ is the product of a linear polynomial and a cubic polynomial, how many roots does $P(x) = 0$ have? Explain.
- GET ORGANIZED** Copy and complete the graphic organizer. Give an example of a polynomial with each type of root.



Rational	Irrational
Polynomial Roots	
Real	Complex

GUIDED PRACTICE

SEE EXAMPLE 1

p. 445

Write the simplest polynomial function with the given zeros.

1. $\frac{1}{3}, 1, 2$

2. $-2, 2, 3$

3. $-2, \frac{1}{2}, 2$

SEE EXAMPLE 2

p. 446

Solve each equation by finding all roots.

4. $x^4 - 81 = 0$

5. $3x^3 - 10x^2 + 10x - 4 = 0$

6. $x^3 - 3x^2 + 4x - 12 = 0$

SEE EXAMPLE 3

p. 447

Write the simplest polynomial function with the given zeros.

7. $1 - i$ and 2

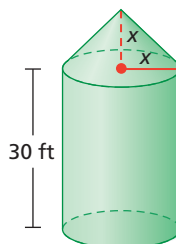
8. $1 + \sqrt{5}$ and 3

9. $2i, \sqrt{2}$, and 2

SEE EXAMPLE 4

p. 447

10. **Farming** A grain silo is shaped like a cylinder with a cone-shaped top. The cylinder is 30 feet tall. The volume of the silo is 1152π cubic feet. Find the radius of the silo.



PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
11–13	1
14–19	2
20–22	3
23	4

Write the simplest polynomial function with the given zeros.

11. $-1, -1, 2$

12. $2, 1, \frac{2}{3}$

13. $-4, -1, 2$

Solve each equation by finding all roots.

14. $x^4 - 16 = 0$

15. $x^3 - 7x^2 + 15x - 9 = 0$

16. $x^4 + 5x^2 - 36 = 0$

17. $2x^3 - 3x^2 + 8x - 12 = 0$

18. $x^4 - 5x^3 + 3x^2 + x = 0$

19. $x^4 - 4x^2 + 3 = 0$

Extra Practice

Skills Practice p. S15

Application Practice p. S37

Write the simplest polynomial function with the given zeros.

20. $2 - i, \sqrt{3}$, and 2

21. $2\sqrt{2}, \sqrt{5}$, and -3

22. $-2i$ and $1 + i$

23. **Storage** A storage bin is shaped like a cylinder with a hemisphere-shaped top. The cylinder is 45 inches tall. The volume of the bin is 4131π cubic inches. Find the radius of the bin.

Solve each equation by finding all roots.

24. $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$

25. $x^4 + 4x^3 - 3x^2 - 14x - 8 = 0$

26. $x^3 + 3x^2 + 3x + 1 = 0$

27. $x^4 + 4x^3 + 6x^2 + 4x + 1 = 0$

28. $6x^3 + 11x^2 - 3x - 2 = 0$

29. $x^3 - 2x^2 - 2x - 3 = 0$

30. $x^3 - 6x^2 + 11x - 6 = 0$

31. $x^4 - 13x^3 + 55x^2 - 91x = 0$

32. $x^4 + x^2 - 12 = 0$

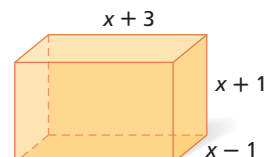
33. $x^4 + 14x^2 + 45 = 0$

34. $x^3 + 13x - 85 = 31$

35. $x^3 - 4x^2 + x + 14 = 8$



36. **Geometry** The volume of the rectangular prism is 105 cubic units. Find the dimensions of the prism.





37. This problem will prepare you for the Multi-Step Test Prep on page 472.

The volume of a pyramid-shaped tent with a square base can be represented by the function $V(x) = \frac{1}{3}x^3 - 2x^2$, where x is the length of the base in meters.

- The volume of the tent is 81 m^3 . Write a polynomial equation with integer coefficients that you can solve in order to find the length of the base.
- Find the length of the base.
- What can you say about the other roots of the polynomial equation? Why?

Write the simplest polynomial function with the given zeros.

38. $0, \sqrt{5}$, and 2 39. $4i, 2$, and -2 40. $1, -1$ (multiplicity of 3), and $3i$
41. $1, 1$, and 2 42. $1 - \sqrt{2}$, and $2i$ 43. 3 (multiplicity of 2), and $3i$

Tell whether each statement is sometimes, always, or never true. If it is sometimes true, give examples to support your answer.

- A cubic polynomial has no real zeros.
- A quartic polynomial has an odd number of real zeros.
- There are infinitely many polynomials with zeros a, b , and c .
- The multiplicity of a root is equal to the degree of the polynomial.

Use your graphing calculator to approximate the solutions of each equation by finding all roots. Round your answer to the nearest thousandth.

48. $3x^4 - x = 6x^2 + \sqrt{2}$ 49. $2\sqrt{3}x^4 - x^2 = 0$
50. $-6x^3 = -5\sqrt{3}x + \sqrt{2}$ 51. $\sqrt{7}x^3 - 11x^2 + 8 = 0$

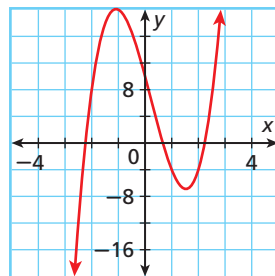
52. **Forestry** The volume of a giant sequoia can be modeled by $V(h) = 0.485h^3 - 362h^2 + 89889h - 7379874$, where h represents the height of a tree in feet and $220 \leq h \leq 280$.

- Find the height of a tree with volume 39,186 cubic feet.
- The Lincoln tree in the Giant Forest in Sequoia National Park has a volume of 44,471 cubic feet. What are its possible heights?
- The actual height of the Lincoln tree is 255.8 feet. What is the difference between the true volume of the tree and the volume given by the model?

53. The volume of a cylindrical propane tank with a hemispherical top and bottom can be represented by the function $V(r) = 33\pi r^2 + \frac{4}{3}\pi r^3$, where V is the volume in cubic inches and r is the radius in inches. What is the radius if the volume of the tank is 1476π cubic inches?

54. **Critical Thinking** What is the least degree of a polynomial equation that has $3i$ as a root with a multiplicity of 3? Explain.

55. **Estimation** Use the graph to estimate the roots of $y = 3x^3 - 2x^2 - 15x + 10$. Then find the exact roots. (Hint: Factor by grouping.)



56. **Write About It** Describe your method for factoring a fourth-degree polynomial. What are the different situations that you need to consider?



Forestry



The sequoias in Sequoia National Park in California are among the largest trees in the world. Many are as tall as a 26-story building and wider than some city streets.

57. What is the multiplicity of the root -1 in the equation $y = x^4 - 2x^3 - 3x^2 + 4x + 4$?
 (A) 1 (B) 2 (C) 3 (D) 4
58. What is the conjugate of $-6 - 5i$?
 (F) $-6 - 5i$ (G) $-6 + 5i$ (H) $6 - 5i$ (J) $6 + 5i$
59. Which polynomial function has zeros 0 , i , and $-i$?
 (A) $P(x) = x^3 + 2x^2 + 1$ (C) $P(x) = x^3 + 2x^2 + x$
 (B) $P(x) = x^3 + x^2$ (D) $P(x) = x^3 + x$
60. A polynomial has zeros $3 - \sqrt{2}$, 4 , and $6i$. What is the minimum degree of the polynomial?
 (F) 3 (G) 4 (H) 5 (J) 6
61. Which polynomial function has zeros $1 + \sqrt{3}$ and $1 - \sqrt{3}$?
 (A) $f(x) = x^2 + 8$ (C) $f(x) = x^2 - x + \sqrt{3}$
 (B) $f(x) = x^2 - 3x + 9$ (D) $f(x) = x^2 - 2x - 2$
62. **Short Response** Solve the equation $2x^3 - 6x^2 + 8x - 24 = 0$ by finding all roots.

CHALLENGE AND EXTEND

63. Use synthetic substitution to evaluate $f(x) = x^3 - 3x^2 + 9x - 27$ for $x = 3i$ and $x = -\sqrt{3}$. Is either $x = 3i$ or $x = -\sqrt{3}$ a zero of f ?
64. $2i$ is a zero of $P(x) = x^3 - 2ix^2 - 4x + 8i$. Find the other zeros of P .
65. One zero of $Q(x) = x^3 - \sqrt{2}x^2 + 9x - 9\sqrt{2}$ is $\sqrt{2}$. Find the other zeros of Q .
66. **Critical Thinking** Based on your answers to Exercises 64 and 65, what can you say about a polynomial function with nonreal or irrational coefficients?
67. Factor the sum $a^2 + b^2$.
68. Factor the sum $a^4 + b^4$.
69. Factor the sum $a^6 + b^6$.
70. **Critical Thinking** Does your answer to Exercise 67 help you answer Exercises 68 or 69? Explain.
71. **Critical Thinking** Give an example of a fourth-degree polynomial equation that has no real zeros. What are the roots of your example?

SPIRAL REVIEW

State the transformation that maps the graph of $f(x) = |x|$ onto the graph of each function. Then graph each function. (Lesson 2-9)

72. $f(x) = |x + 5| - 7$

73. $f(x) = -|x - 2| + 3$

Find the vertex of the graph of each function. (Lesson 2-9)

74. $g(x) = |x + 3| - 4$

75. $f(x) = |x - 9| - 3$

Solve each equation by factoring. (Lesson 6-5)

76. $4x^4 + 32x^3 + 64x^2 = 0$

77. $x^3 - 43x^2 + 42x = 0$

78. $3x^5 + 18x^4 - 81x^3 = 0$

79. $2x^3 + 12x^2 = 32x$



Explore End Behavior

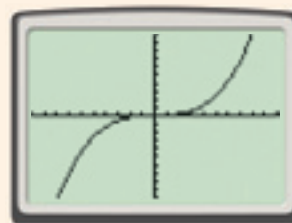
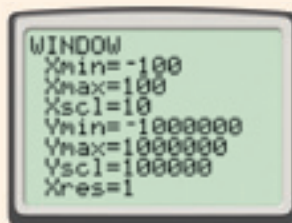
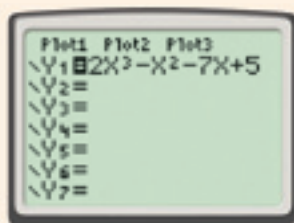
End behavior is a description of the values of the function as x approaches positive infinity ($x \rightarrow +\infty$) or negative infinity ($x \rightarrow -\infty$).

Use with Lesson 6-7

Activity

Describe the end behavior of $f(x) = 2x^3 - x^2 - 7x + 5$.

Enter $f(x)$ into your graphing calculator. Choose a large window, and graph.



Notice $f(x)$ appears to rise for positive x -values and fall for negative x -values.

Try This

- Consider the functions $g(x) = 3x^3 - 2x^2 + x + 4$, $h(x) = \frac{1}{2}x^3 + 3x^2 + x - 9$, and $k(x) = 5x^3 - 8x^2 - 2x + 1$.
 - What do the functions $g(x)$, $h(x)$, and $k(x)$ have in common?
 - Graph $g(x)$, $h(x)$, and $k(x)$ on your graphing calculator, and describe the end behavior of each.
 - Make a Conjecture** What can you say about the end behavior of functions of the same type as $g(x)$, $h(x)$, and $k(x)$?
- Consider the functions $a(x) = -3x^3 - 2x^2 + x + 4$, $b(x) = -\frac{1}{2}x^3 + 3x^2 + x - 9$, and $c(x) = -5x^3 - 8x^2 - 2x + 1$.
 - What do the functions $a(x)$, $b(x)$, and $c(x)$ have in common?
 - Graph $a(x)$, $b(x)$, and $c(x)$ on your graphing calculator, and describe the end behavior of each.
 - Make a Conjecture** What can you say about the end behavior of functions of the same type as $a(x)$, $b(x)$, and $c(x)$?
- Consider the functions $p(x) = 3x^4 - x^2 + x + 4$, $r(x) = \frac{1}{2}x^4 + 3x^3 + x - 9$, and $s(x) = 5x^4 - 8x^3 - 2x^2 + 1$.
 - What do the functions $p(x)$, $r(x)$, and $s(x)$ have in common?
 - Graph $p(x)$, $r(x)$, and $s(x)$ on your graphing calculator, and describe the end behavior of each.
 - Make a Conjecture** What can you say about the end behavior of functions of the same type as $p(x)$, $r(x)$, and $s(x)$?
- Critical Thinking** Compare your conjectures from Problems 1c, 2c, and 3c. What are the characteristics of a function that seem to affect the function's end behavior?





6-7

Investigating Graphs of Polynomial Functions



Objectives

Use properties of end behavior to analyze, describe, and graph polynomial functions.

Identify and use maxima and minima of polynomial functions to solve problems.

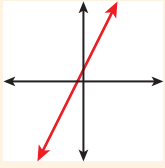
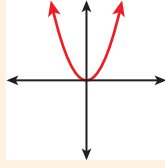
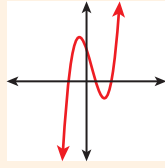
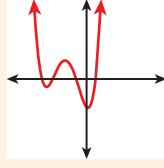
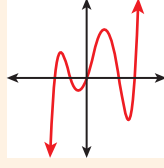
Vocabulary

end behavior
turning point
local maximum
local minimum

Who uses this?

Welders can use graphs of polynomial functions to optimize the use of construction materials. (See Example 5.)

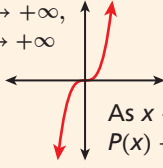
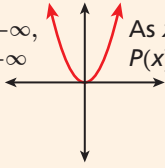
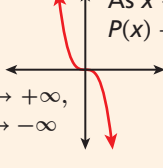
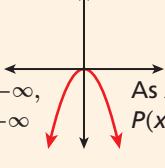
Polynomial functions are classified by their degree. The graphs of polynomial functions are classified by the degree of the polynomial. Each graph, based on the degree, has a distinctive shape and characteristics.

Graphs of Polynomial Functions				
Linear function Degree 1	Quadratic function Degree 2	Cubic function Degree 3	Quartic function Degree 4	Quintic function Degree 5
				

End behavior is a description of the values of the function as x approaches positive infinity ($x \rightarrow +\infty$) or negative infinity ($x \rightarrow -\infty$). The degree and leading coefficient of a polynomial function determine its end behavior. It is helpful when you are graphing a polynomial function to know about the end behavior of the function.



Polynomial End Behavior

$P(x)$ has...	Odd Degree	Even Degree
Leading coefficient $a > 0$	As $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$  As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$	As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$  As $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$
Leading coefficient $a < 0$	As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$  As $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$	As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$  As $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$

EXAMPLE 1 Determining End Behavior of Polynomial Functions

Helpful Hint

Both the leading coefficient and the degree of the polynomial are contained in the term of greatest degree. When determining end behavior, you can ignore all other terms.

Identify the leading coefficient, degree, and end behavior.

A $P(x) = -4x^3 - 3x^2 + 5x + 6$

The leading coefficient is -4 , which is negative.

The degree is 3 , which is odd.

As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$.

B $R(x) = x^6 - 7x^5 + x^3 - 2$

The leading coefficient is 1 , which is positive.

The degree is 6 , which is even.

As $x \rightarrow -\infty$, $P(x) \rightarrow +\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$.



Identify the leading coefficient, degree, and end behavior.

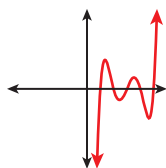
1a. $P(x) = 2x^5 + 3x^2 - 4x - 1$

1b. $S(x) = -3x^2 + x + 1$

EXAMPLE 2 Using Graphs to Analyze Polynomial Functions

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

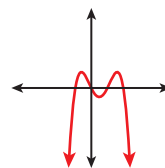
A



As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and
as $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$.

$P(x)$ is of odd degree with a
positive leading coefficient.

B



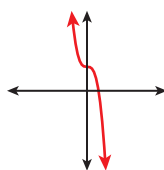
As $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and
as $x \rightarrow +\infty$, $P(x) \rightarrow -\infty$.

$P(x)$ is of even degree with a
negative leading coefficient.

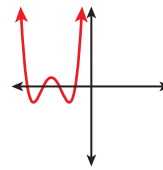


Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

2a.



2b.



Now that you have studied factoring, solving polynomial equations, and end behavior, you can graph a polynomial function.

Steps for Graphing a Polynomial Function

1. Find the real zeros and y-intercept of the function.
2. Plot the x- and y-intercepts.
3. Make a table for several x-values that lie between the real zeros.
4. Plot the points from your table.
5. Determine the end behavior of the graph.
6. Sketch the graph.

EXAMPLE 3 Graphing Polynomial Functions

Graph the function.

$$f(x) = x^3 + 3x^2 - 6x - 8$$

Step 1 Identify the possible rational roots by using the Rational Root Theorem.

$$\pm 1, \pm 2, \pm 4, \pm 8 \quad p = -8 \text{ and } q = 1$$

Step 2 Test possible rational zeros until a zero is identified.

Test $x = 1$.

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -6 & -8 \\ & & 1 & 4 & -2 \\ \hline & 1 & 4 & -2 & -10 \end{array}$$

Test $x = -1$.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & 0 \end{array}$$

$x = -1$ is a zero, and $f(x) = (x + 1)(x^2 + 2x - 8)$.

Step 3 Factor: $f(x) = (x + 1)(x - 2)(x + 4)$.

The zeros are -1 , 2 , and -4 .

Step 4 Plot other points as guidelines.

$f(0) = -8$, so the y -intercept is -8 .

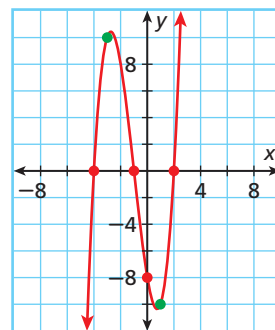
Plot points between the zeros. Choose $x = -3$ and $x = 1$ for simple calculations.

$$f(-3) = 10 \text{ and } f(1) = -10$$

Step 5 Identify end behavior.

The degree is odd and the leading coefficient is positive so as $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$, and as $x \rightarrow +\infty$, $P(x) \rightarrow +\infty$.

Step 6 Sketch the graph of $f(x) = x^3 + 3x^2 - 6x - 8$ by using all of the information about $f(x)$.



Graph each function.

3a. $f(x) = x^3 - 2x^2 - 5x + 6$

3b. $f(x) = -2x^2 - x + 6$

A **turning point** is where a graph changes from increasing to decreasing or from decreasing to increasing. A turning point corresponds to a *local maximum* or *minimum*.

Local Maxima and Minima

For a function $f(x)$, $f(a)$ is a **local maximum** if there is an interval around a such that $f(x) < f(a)$ for every x -value in the interval except a .

For a function $f(x)$, $f(a)$ is a **local minimum** if there is an interval around a such that $f(x) > f(a)$ for every x -value in the interval except a .

A polynomial function of degree n has at most $n - 1$ turning points and at most n x -intercepts. If the function has n distinct real roots, then it has exactly $n - 1$ turning points and exactly n x -intercepts. You can use a graphing calculator to graph and estimate maximum and minimum values.

EXAMPLE 4 Determine Maxima and Minima with a Calculator

Graph $g(x) = 2x^3 - 12x + 6$ on a calculator, and estimate the local maxima and minima.

Step 1 Graph.

The graph appears to have one local maximum and one local minimum.

Step 2 Find the maximum.

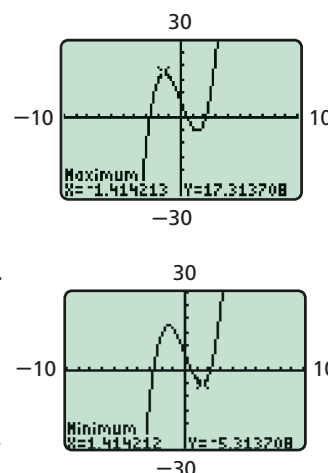
Press **2nd** **TRACE** to access the **CALC** menu. Choose **4:maximum**.

The local maximum is approximately 17.3137.

Step 3 Find the minimum.

Press **2nd** **TRACE** to access the **CALC** menu. Choose **3:minimum**.

The local minimum is approximately -5.3137.



Reading Math

Maxima is the plural form of *maximum*.
Minima is the plural form of *minimum*.



Graph each function on a calculator, and estimate the local maxima and minima.

4a. $g(x) = x^3 - 2x - 3$

4b. $h(x) = x^4 + 4x^2 - 6$

EXAMPLE 5 Industrial Application

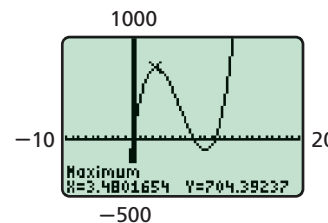
A welder plans to construct an open box from an 18.5 ft by 24.5 ft sheet of metal by cutting squares from the corners and folding up the sides. Find the maximum volume of the box and the corresponding dimensions.

Find a formula to represent volume.

$$V(x) = x(18.5 - 2x)(24.5 - 2x) \quad V = \ell wh$$

Graph $V(x)$. Note that values of x greater than 9.25 or less than 0 do not make sense for this problem.

The graph has a local maximum of about 704.4 when $x \approx 3.48$. So, the largest open box will have a volume of 704.4 ft^3 .



5. What is the maximum volume of a box made from a 16 ft by 20 ft sheet of metal?

THINK AND DISCUSS

1. Explain why a polynomial function that has exactly n distinct real roots must have $n - 1$ turning points.
2. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, sketch a graph that fits the description.

Leading Coefficient	Odd Degree	Even Degree
Positive		
Negative		



GUIDED PRACTICE

1. **Vocabulary** Explain why a *turning point* is appropriately named.

SEE EXAMPLE 1

p. 454

- Identify the leading coefficient, degree, and end behavior.

2. $P(x) = -4x^4 - 3x^3 + x^2 + 4$

3. $Q(x) = -2x^7 + 6x^5 + 2x^3$

4. $R(x) = x^5 - 4x^2 + 3x - 1$

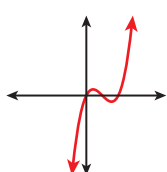
5. $S(x) = 3x^2 + 6x - 10$

SEE EXAMPLE 2

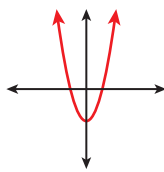
p. 454

- Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

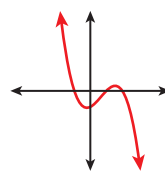
6.



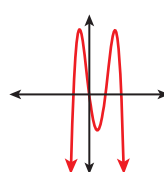
7.



8.



9.



SEE EXAMPLE 3

p. 455

- Graph each function.

10. $f(x) = x^2 - 5x - 50$

11. $f(x) = -x^3 + \frac{3}{2}x^2 + 25x + 12$

SEE EXAMPLE 4

p. 456

- Graph each function on a calculator, and estimate the local maxima and minima.

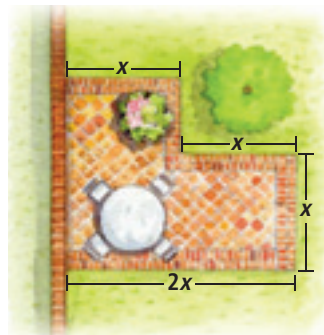
12. $f(x) = x^4 - 4x^3 + 3x + 5$

13. $f(x) = 2x^3 - 3x^2 - 6x - 5$

SEE EXAMPLE 5

p. 456

14. **Landscape Design** Vera has 60 ft of fencing and wants to enclose a patio, using an existing wall for one side as shown. The area of the patio can be modeled by $A(x) = 60x - 3x^2$, where x is in feet. Find the maximum area of the patio.



PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
15–18	1
19–22	2
23–26	3
27–30	4
31	5

Extra Practice

Skills Practice p. S15

Application Practice p. S37

- Identify the leading coefficient, degree, and end behavior.

15. $P(x) = 2x^3 + 3x^2 - 4x$

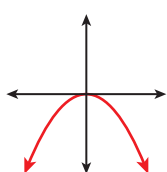
16. $Q(x) = -3x^4 - 8x^2$

17. $R(x) = -x^5 + 5x^4 + 1$

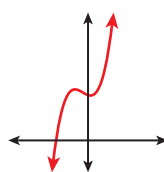
18. $S(x) = 5.5x^8 + 7.5x^4$

- Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

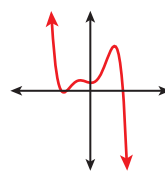
19.



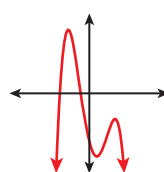
20.



21.



22.



- Graph each function.

23. $f(x) = x^3 - \frac{7}{3}x^2 - \frac{43}{3}x + 5$

24. $f(x) = 25x^2 - 4$

25. $f(x) = x^4 + x^3 - 28x^2 + 20x + 48$

26. $f(x) = x^3 + \frac{13}{2}x^2 + 11x + 4$



Health



A person's total lung capacity depends on many factors. Typically, age, weight, gender, and physical fitness affect the volume of air that a person's lungs can hold.

Graph each function on a calculator, and estimate the local maxima and minima.

27. $f(x) = 9x^6 + 20$

28. $f(x) = x^3 - 4x^2 + x + 1$

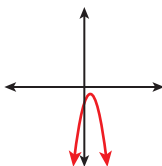
29. $f(x) = -x^2 + 6x - 10$

30. $f(x) = -5x^2 + 7$

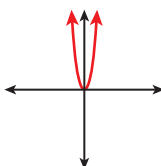
31. **Health** The volume of air (in liters) in the human lung during one normal breath can be modeled by the function $V(t) = -1.7t^2 + 1.7t + 3$, for $0 \leq t \leq 1$. What is the maximum volume of air in the lungs during a normal breath, and at what time does it occur?

Use the degree and end behavior to match each polynomial to its graph.

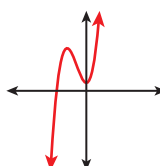
A.



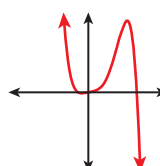
B.



C.



D.



32. $5x^3 + 9x^2 + 1$

33. $2x^6 + 3x^4 + 5x^2$

34. $3x^4 - x^5 + x$

35. $-4x^2 + 3x - 1$

Describe the end behavior of each function by completing the statements

$f(x) \rightarrow$ _____ as $x \rightarrow -\infty$, and $f(x) \rightarrow$ _____ as $x \rightarrow +\infty$.

36. $f(x) = 2x^5 - x^2 + 75$

37. $f(x) = 10x^4 + 9x^2$

38. $f(x) = -5x^3 + x - 8$

39. $f(x) = 1000x^4 - 0.0002x^8$

40. $f(x) = x^{13} - x^7 + 12x$

41. $f(x) = -331x^{44} + 98$

42. **Retail** Hiromi sells 12 T-shirts each week at a price of \$13.00. Past sales have shown that for every \$0.25 decrease in price, 4 more T-shirts are sold. Knowing that revenue is a product of price and quantity, Hiromi models his revenue by $R(x) = (13 - 0.25x)(12 + 4x)$, where x represents the number of times there is a reduction in price.
- Graph the function on a graphing calculator.
 - What is the maximum revenue Hiromi can generate each week?
 - How many \$0.25 reductions will maximize Hiromi's revenue? What would be the price per T-shirt, given the price is $13 - 0.25x$?
43. **Critical Thinking** Why are the leading coefficient and degree of the first term of the polynomial the only characteristics that determine end behavior?
44. Which of the following functions has end behavior that is different from the others? Explain.
 $f(x) = 2x^7 - 10x^4 + 125$ $g(x) = -x^7 + x^6$ $h(x) = 5x^3 - 1$ $k(x) = -x + 4x^5$

MULTI-STEP TEST PREP



45. This problem will prepare you for the Multi-Step Test Prep on page 472.

A packaging company wants to manufacture a pyramid-shaped gift box with a rectangular base. The base must have a perimeter of 20 in., and the height of the box must be equal to the length of the base.

- Write a polynomial function, $V(x)$, for the volume of the box, where x is the length of the base.
- Find the maximum volume of the box.
- What dimensions result in a box with the maximum volume?

46. **Critical Thinking** Is there always an x -intercept between two turning points? Explain.



47. **Write About It** Describe the steps for graphing a polynomial by hand.



48. How many turning points will a quartic function with four real zeros have?

(A) 1 (B) 2 (C) 3 (D) 4

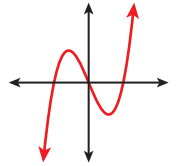
49. Which function could describe the graph?

(F) $f(x) = -2x^5 + x - 4$

(H) $f(x) = 3x^3 - 9x$

(G) $f(x) = -x^3 + 5x^2 + 4x + 3$

(J) $f(x) = \frac{1}{4}x^2 + \frac{1}{2}x + 1$



50. **Extended Response** Consider the polynomial function

$$f(x) = 2x^4 + 12x^3 + 24x^2 + 16x.$$

- Find all solutions to the equation $f(x) = 0$.
- Describe the end behavior of $f(x)$. Explain your reasoning.
- Sketch a graph of $f(x)$ by using your answers to part a and part b.

CHALLENGE AND EXTEND

Graph each function without using a graphing calculator.

51. $3x^6 - 57x^4 + 6x^3 + 144x^2 - 96x$

52. $-2x^5 - 14x^4 - 30x^3 - 18x^2$

Examine the behavior of the cubic polynomials $f(x) = x^3$ and $g(x) = x^3 - 6x^2 + 4x - 20$ for the given values of x by using a spreadsheet or graphing calculator.

53. Complete the table.

54. Use your answer to Exercise 50 to complete this statement.

As $x \rightarrow +\infty$, $\frac{f(x)}{g(x)} \rightarrow \underline{\hspace{2cm}}$.

55. Explain what the statement in Exercise 54 implies about the end behavior of $f(x)$ and $g(x)$.

x	$f(x)$	$g(x)$	$\frac{f(x)}{g(x)}$
5	■	■	■
10	■	■	■
50	■	■	■
100	■	■	■
500	■	■	■
1000	■	■	■
5000	■	■	■

SPIRAL REVIEW

Use substitution to determine if the given point is a solution to the system of equations. (*Lesson 3-1*)

56. $(2, -2) \begin{cases} 2x + y = 2 \\ 6x - 2y = 16 \end{cases}$

57. $(3, -1) \begin{cases} x + 3y = 0 \\ 8x + 4y = 21 \end{cases}$

58. $(5, -5) \begin{cases} x + y = 5 \\ x - y = 10 \end{cases}$

59. **Sports** A tennis ball is served with an initial velocity of 64 ft/s^2 from 6 ft above the ground. The tennis ball's height in feet is modeled by $h(t) = -16t^2 + 64t + 6$, where t is the time in seconds after the tennis ball is hit. Write the function in vertex form and identify the vertex. (*Lesson 5-4*)

Divide by using long division. (*Lesson 6-3*)

60. $x + 3 \overline{)x^2 + 4x + 10}$

61. $x + 1 \overline{)10x^2 + 8x + 6}$

62. $x + 8 \overline{)x^2 + x - 64}$



6-8

Transforming Polynomial Functions



Objective

Transform polynomial functions.

Why learn this?

Transformations can be used in business to model sales. (See Example 5.)

You can perform the same transformations on polynomial functions that you performed on quadratic and linear functions.



Transformations of $f(x)$		
Transformation	$f(x)$ Notation	Examples
Vertical translation	$f(x) + k$	$g(x) = x^3 + 3$ 3 units up $g(x) = x^3 - 4$ 4 units down
Horizontal translation	$f(x - h)$	$g(x) = (x - 2)^3$ 2 units right $g(x) = (x + 1)^3$ 1 unit left
Vertical stretch/compression	$af(x)$	$g(x) = 6x^3$ stretch by 6 $g(x) = \frac{1}{2}x^3$ compression by $\frac{1}{2}$
Horizontal stretch/compression	$f\left(\frac{1}{b}x\right)$	$g(x) = \left(\frac{1}{5}x\right)^3$ stretch by 5 $g(x) = (3x)^3$ compression by $\frac{1}{3}$
Reflection	$-f(x)$ $f(-x)$	$g(x) = -x^3$ across x -axis $g(x) = (-x)^3$ across y -axis

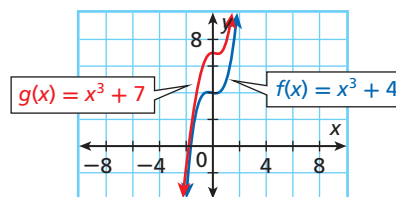
EXAMPLE 1 Translating a Polynomial Function

For $f(x) = x^3 + 4$, write the rule for each function and sketch its graph.

A $g(x) = f(x) + 3$
 $g(x) = (x^3 + 4) + 3$
 $g(x) = x^3 + 7$

To graph $g(x) = f(x) + 3$, translate the graph of $f(x)$ 3 units up.

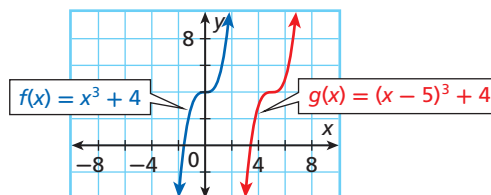
This is a vertical translation.



B $g(x) = f(x - 5)$
 $g(x) = (x - 5)^3 + 4$
 $g(x) = (x - 5)^3 + 4$

To graph $g(x) = f(x - 5)$, translate the graph of $f(x)$ 5 units right.

This is a horizontal translation.



Helpful Hint

You can use a calculator to check your graph.



For $f(x) = x^3 + 4$, write the rule for each function and sketch its graph.

1a. $g(x) = f(x) - 5$

1b. $g(x) = f(x + 2)$

EXAMPLE 2 Reflecting Polynomial Functions

Let $f(x) = x^3 - 7x^2 + 6x - 5$. Write a function g that performs each transformation.

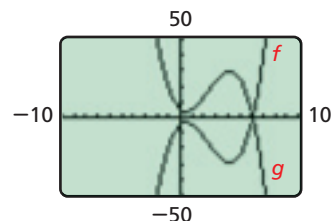
A Reflect $f(x)$ across the x -axis.

$$g(x) = -f(x)$$

$$g(x) = -(x^3 - 7x^2 + 6x - 5)$$

$$g(x) = -x^3 + 7x^2 - 6x + 5$$

Check Graph both functions. The graph appears to be a reflection. ✓



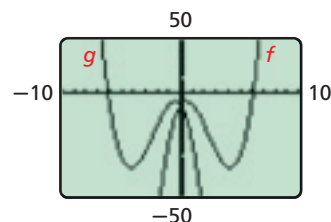
B Reflect $f(x)$ across the y -axis.

$$g(x) = f(-x)$$

$$g(x) = (-x)^3 - 7(-x)^2 + 6(-x) - 5$$

$$g(x) = -x^3 - 7x^2 - 6x - 5$$

Check Graph both functions. The graph appears to be a reflection. ✓



Let $f(x) = x^3 - 2x^2 - x + 2$. Write a function $g(x)$ that performs each transformation.

2a. Reflect $f(x)$ across the x -axis.

2b. Reflect $f(x)$ across the y -axis.

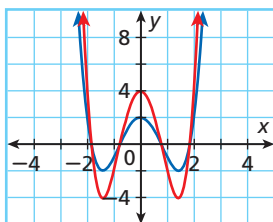
EXAMPLE 3 Compressing and Stretching Polynomial Functions

Let $f(x) = x^4 - 4x^2 + 2$. Graph f and g on the same coordinate plane. Describe g as a transformation of f .

A $g(x) = 2f(x)$

$$g(x) = 2(x^4 - 4x^2 + 2)$$

$$g(x) = 2x^4 - 8x^2 + 4$$

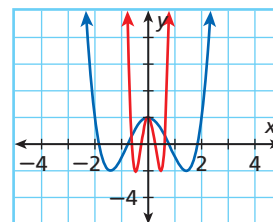


$g(x)$ is a vertical stretch of $f(x)$.

B $g(x) = f(3x)$

$$g(x) = (3x)^4 - 4(3x)^2 + 2$$

$$g(x) = 81x^4 - 36x^2 + 2$$



$g(x)$ is a horizontal compression of $f(x)$.



Let $f(x) = 16x^4 - 24x^2 + 4$. Graph f and g on the same coordinate plane. Describe g as a transformation of f .

3a. $g(x) = \frac{1}{4}f(x)$

3b. $g(x) = f\left(\frac{1}{2}x\right)$

EXAMPLE 4 Combining Transformations

Write a function that transforms $f(x) = 3x^3 + 6$ in each of the following ways. Support your solution by using a graphing calculator.

- A** Stretch vertically by a factor of 2, and shift 3 units left.

A vertical stretch is represented by $af(x)$, and a horizontal shift is represented by $f(x - h)$.

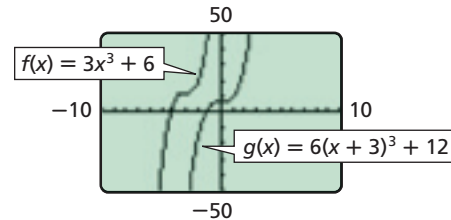
Combining the two transformations gives $g(x) = af(x - h)$.

Substitute 2 for a and 3 for h .

$$g(x) = 2f(x + 3)$$

$$g(x) = 2(3(x + 3)^3 + 6)$$

$$g(x) = 6(x + 3)^3 + 12$$



- B** Reflect across the x -axis and shift 3 units up.

A reflection across the x -axis is represented by $-f(x)$, and a vertical shift is represented by $f(x) + k$. Combining the two transformations gives

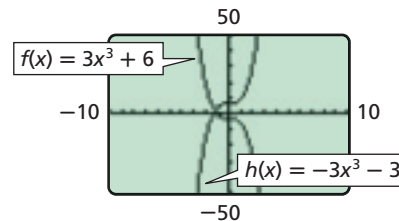
$$h(x) = -f(x) + k.$$

Substitute 3 for k .

$$h(x) = -f(x) + 3$$

$$h(x) = -(3x^3 + 6) + 3$$

$$h(x) = -3x^3 - 3$$



Write a function that transforms $f(x) = 8x^3 - 2$ in each of the following ways. Support your solution by using a graphing calculator.

- 4a. Compress vertically by a factor of $\frac{1}{2}$, and move the x -intercept 3 units right.
4b. Reflect across the x -axis, and move the x -intercept 4 units left.

EXAMPLE 5 Bicycle Sales

The number of bicycles sold per month by a business can be modeled by $f(x) = 0.01x^3 + 0.7x^2 + 0.4x + 120$, where x represents the number of months since January. Let $g(x) = f(x) - 30$. Find the rule for g , and explain the meaning of the transformation in terms of monthly bicycle sales.

Step 1 Write the new rule.

The new rule is $g(x) = f(x) - 30$.

$$g(x) = f(x) - 30$$

$$g(x) = 0.01x^3 + 0.7x^2 + 0.4x + 120 - 30$$

$$g(x) = 0.01x^3 + 0.7x^2 + 0.4x + 90$$

Step 2 Interpret the transformation.

The transformation represents a vertical shift 30 units down, which corresponds to a decrease in sales of 30 units per month.



5. Let $g(x) = f(x - 5)$. Find the rule for g , and explain the meaning of the transformation in terms of monthly bicycle sales.

THINK AND DISCUSS

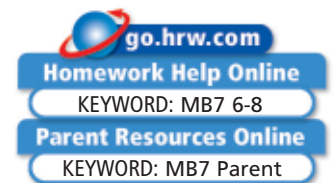
1. How does shifting $f(x) = x^4 - 4$ up 5 units affect the number of real zeros? What if $f(x)$ is shifted 5 units down?
2. Does a horizontal shift affect the number of real zeros of a function? Explain.
3. **GET ORGANIZED** Copy and complete the graphic organizer.



Transformation	Vertical shift	Horizontal shift	Vertical stretch	Horizontal compression
Example				

6-8

Exercises



GUIDED PRACTICE

SEE EXAMPLE

1

For $f(x) = x^4 - 8$, write the rule for each function, and sketch its graph.

p. 460

1. $g(x) = f(x) + 4$
2. $h(x) = f(x - 2)$
3. $j(x) = f(3x)$
4. $k(x) = f(x) - \frac{1}{2}$

SEE EXAMPLE

2

Let $f(x) = -x^3 + 3x^2 - 2x + 1$. Write a function g that performs each transformation.

p. 461

5. Reflect $f(x)$ across the y -axis.
6. Reflect $f(x)$ across the x -axis.

SEE EXAMPLE

3

Let $f(x) = x^3 - 4x^2 + 2$. Graph f and g on the same coordinate plane. Describe g as a transformation of f .

p. 461

7. $g(x) = f\left(\frac{1}{2}x\right)$
8. $g(x) = 3f(x)$
9. $g(x) = f(2x) + 4$

SEE EXAMPLE

4

Write a function that transforms $f(x) = 4x^3 + 2$ in each of the following ways. Support your solution by using a graphing calculator.

p. 462

10. Compress vertically by a factor of $\frac{1}{2}$, and move the y -intercept 2 units down.
11. Reflect across the y -axis, and compress horizontally by a factor of $\frac{1}{2}$.
12. Move 2 units right, move 3 units down, and reflect across the x -axis.

SEE EXAMPLE

5

13. Manufacturing The cost to manufacture x units of a product can be modeled by the function $C(x) = 2x^3 - 3x + 30$, where the cost is in thousands of dollars. Describe the transformation $2C(x)$ by writing the new rule and explaining the change in the context of the problem.

p. 462

PRACTICE AND PROBLEM SOLVING

For $f(x) = x^3 - 4$, write the rule for each function and sketch its graph.

14. $g(x) = f(x) - 3$
15. $h(x) = f(x - 3)$
16. $j(x) = f(x) + 5$

Let $f(x) = x^3 - 2x^2 + 5x - 3$. Write a function g that performs each transformation.

17. Reflect $f(x)$ across the x -axis.
18. Reflect $f(x)$ across the y -axis.

Independent Practice

For Exercises	See Example
14–16	1
17–18	2
19–21	3
22–24	4
25	5

Extra Practice

Skills Practice: S15

Application Practice: S37

Let $f(x) = 2x^4 - 8x^2 - 2$. Graph f and g on the same coordinate plane. Describe g as a transformation of f .

19. $g(x) = 2f(x)$

20. $g(x) = \frac{1}{2}f(x)$

21. $g(x) = f\left(\frac{1}{2}x\right)$

Write a function that transforms $f(x) = x^4 - 6$ in each of the following ways. Support your solution by using a graphing calculator.

22. Reflect across the x -axis, and move the x -intercept 3 units left.

23. Compress vertically by a factor of $\frac{1}{3}$, and move 1 unit up.

24. Stretch horizontally by a factor of 2, move 4 units down, and reflect across the y -axis.



25. **Geometry** The volume of a rectangular prism can be modeled by the function $V(x) = x^3 + 3x^2 + x + 8$, where V is the volume in cubic meters and x represents length in meters. Describe the transformation $V\left(\frac{2}{3}x\right)$ by writing the new rule and explaining the change in the context of the problem.

26. **ERROR ANALYSIS** Students were asked to write a function g that translates f 3 units to the right. Which answer is incorrect? Identify and explain the error.

A

$$\begin{aligned} f(x) &= x^3 + 1 \\ g(x) &= (x - 3)^3 + 1 \end{aligned}$$

B

$$\begin{aligned} f(x) &= x^3 + 1 \\ g(x) &= (x + 3)^3 + 1 \end{aligned}$$



27. **Fish** Some flying fish travel in the air up to a quarter of a mile by using a lift force F to overcome their weight W while in the air. The lift force is modeled by $F(v) = 0.24v^2$, where $v \geq 0$ is the initial air speed in meters per second.

a. Graph $F(v) = 0.24v^2$. For $W = 1$, find the values of v such that $F > W$.

b. A flying fish swimming with a current leaps out of the water with a speed that is 5 units greater than normal. Write a function $G(v)$ for the lift force. What transformation does this represent?

c. Find the values of v for which $G > W$.

d. $H(v) = 20v^2$ represents the underwater lift force. What transformation of $F(v)$ does this represent?

28. **Critical Thinking** In the function $f(x) = (x + 5)^4 + k$, for which values of k does the function have two real solutions? no real solutions? Explain.



29. **Write About It** Explain in your own words what happens to the graph of a function when you reflect it across the x -axis.

**MULTI-STEP
TEST PREP**



30. This problem will prepare you for the Multi-Step Test Prep on page 472.

The volume of a pyramid with a square base is modeled by the function $V(x) = \frac{1}{3}x^3 + x^2$, where x is the length of the base in inches.

a. Write a new function, $W(x)$, that gives the volume of the pyramid in cubic inches when the length of the base is expressed in feet.

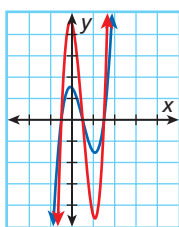
b. Write $W(x)$ in terms of $V(x)$.

c. Graph W and V on the same coordinate plane. How is the graph of W related to the graph of V ?

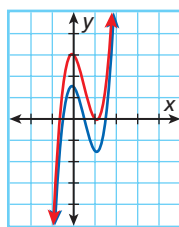
d. How would your answer to part c be different if the length of the base were expressed in centimeters?

31. Which graph represents a vertical shift of $f(x) = x^3 - 3x^2 - x + 3$ up 3 units?

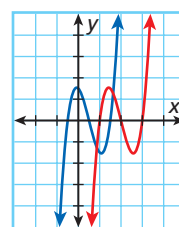
(A)



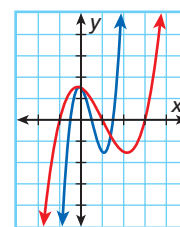
(B)



(C)



(D)



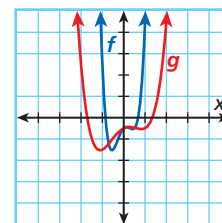
32. Which description matches the transformation from $f(x)$ to $g(x)$ shown?

(F) Vertical shift

(H) Horizontal shift

(G) Vertical stretch

(J) Horizontal stretch



33. $f(x) = x^3 - 6x^2 + 6x + 1$ has three real zeros. How many real zeros does $f(x) - 6$ have?

(A) 0

(C) 2

(B) 1

(D) 3

34. **Extended Response** Consider the function $f(x) = 3x^3 - 9x^2 - 3x + 9$.

a. Use the leading coefficient and degree of $f(x)$ to describe the end behavior.

b. Write the rule for the function $g(x) = f(-x)$, and describe the transformation.

c. Describe the end behavior of $g(x)$. How does the end behavior of $g(x)$ relate to the transformation of $f(x)$?

CHALLENGE AND EXTEND

Identify the transformation(s) that would take $f(x) = (x + 2)^3 - 6$ to $g(x)$.

35. $g(x) = x^3 - 6$

36. $g(x) = (x + 2)^3$

37. $g(x) = (x - 1)^3 + 2$

38. For $f(x) = x^4 - x^2 - 9x + 9$, describe three different transformations that could be performed to obtain a function with a y -intercept of 3.

SPIRAL REVIEW

39. **Computers** An administrative assistant recorded the number of words that he typed in one minute. He then recorded the number of words that he typed during different time intervals. His results are in the table below. Write an equation to express this situation. Is this a function? (Lesson 1-6)

Minutes	1	3	6	8
Words	60	180	360	480

Add or subtract. Write your answer in standard form. (Lesson 6-1)

40. $(6y + 4y^2 - 3) + (9y^2 - 5 + 8y)$

41. $(2x^5 - 4x + 8x^2) - (3x^4 - x^5 + 3x^2)$

Find all of the roots of the polynomial equation. (Lesson 6-6)

42. $x^4 - 81 = 0$

43. $x^4 + x^3 + 3x^2 + 5x - 10 = 0$

44. $x^3 - 5x^2 - 17x + 21 = 0$

45. $x^5 + 3x^4 + 2x^3 + 16x^2 - 48x - 64 = 0$



6-9

Curve Fitting with Polynomial Models



Objectives

Use finite differences to determine the degree of a polynomial that will fit a given set of data.

Use technology to find polynomial models for a given set of data.

Who uses this?

Market analysts can use curve fitting to predict the performance of a stock index. (See Example 3.)

The table shows the closing value of a stock index on the first day of trading for various years.

Year	1994	1995	1996	1997	2000	2001	2003	2004
Price (\$)	774	751	1053	1293	4186	2474	1347	2011

To create a mathematical model for the data, you will need to determine what type of function is most appropriate. In Lesson 5-8, you learned that a set of data that has constant second differences can be modeled by a quadratic function. Finite differences can be used to identify the degree of any polynomial data.



Finite Differences of Polynomials		
Function Type	Degree	Constant Finite Differences
Linear	1	First
Quadratic	2	Second
Cubic	3	Third
Quartic	4	Fourth
Quintic	5	Fifth

EXAMPLE 1 Using Finite Differences to Determine Degree

Use finite differences to determine the degree of the polynomial that best describes the data.

x	-2	-1	0	1	2	3
y	-10	-4	-1.4	0	2.4	8

The x -values increase by a constant 1. Find the differences of the y -values.

y	-10	-4	-1.4	0	2.4	8
---	-----	----	------	---	-----	---

First differences: 6 2.6 1.4 2.4 5.6 **Not constant**
Second differences: -3.4 -1.2 1 3.2 **Not constant**
Third differences: 2.2 2.2 2.2 **Constant**

The third differences are constant. A cubic polynomial best describes the data.

Remember!

To find the differences in the y -values, subtract each y -value from the y -value that follows it. For the first differences, $-4 - (-10) = 6$
 $-1.4 - (-4) = 2.6$
 $0 - (-1.4) = 1.4$ and so on.

Use finite differences to determine the degree of the polynomial that best describes the data.

B

x	-6	-4	-2	0	2	4
y	-30	15	30	34	41	60

The x-values increase by a constant, 2. Find the differences of the y-values.

First differences: 45 15 4 7 19 Not constant
Second differences: -30 -11 3 12 Not constant
Third differences: 19 14 9 Not Constant
Fourth differences: -5 -5 Constant

The fourth differences are constant. A quartic polynomial best describes the data.



1. Use finite differences to determine the degree of the polynomial that best describes the data.

x	12	15	18	21	24	27
y	3	23	29	29	31	43

Once you have determined the degree of the polynomial that best describes the data, you can use your calculator to create the function.

EXAMPLE 2 Using Finite Differences to Write a Function

The table below shows the population of a city from 1950 to 2000. Write a polynomial function for the data.

Year	1950	1960	1970	1980	1990	2000
Population (thousands)	2853	4011	5065	6720	9704	14,759

Helpful Hint

Keep the scale of the original data in mind. In Example 2, the population ranges from 2853 to 14,759. The gap between 705 and 728 is small in comparison.

Step 1 Find the finite differences of the y-values.

Let x represent the number of years since 1950. The years increase by a constant amount of 10. The populations are the y-values.

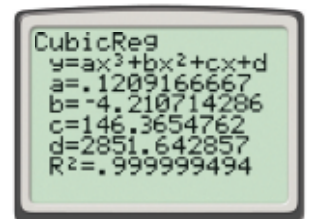
First differences: 1158 1054 1655 2984 5055
Second differences: -104 601 1329 2071
Third differences: 705 728 742 Close

Step 2 Determine the degree of the polynomial.

Because the third differences are relatively close, a cubic function should be a good model.

Step 3 Use the cubic regression feature on your calculator.

$$f(x) \approx 0.12x^3 - 4.21x^2 + 146.37x + 2851.64$$



2. The table below shows the gas consumption of a compact car driven a constant distance at various speeds. Write a polynomial function for the data.

Speed	25	30	35	40	45	50	55	60
Gas (gal)	23.8	25	25.2	25	25.4	27	30.6	37

Often, real-world data can be too irregular for you to use finite differences or find a polynomial function that fits perfectly. In these situations, you can use the regression feature of your graphing calculator. Remember that the closer the R^2 -value is to 1, the better the function fits the data.

EXAMPLE 3 Finance Application

The table shows the opening value of a stock index on the first day of trading in various years. Use a polynomial model to estimate the value on the first day of trading in 2002.

Year	Price (\$)	Year	Price (\$)
1994	774	2000	4186
1995	751	2001	2474
1996	1053	2003	1347
1997	1293	2004	2011

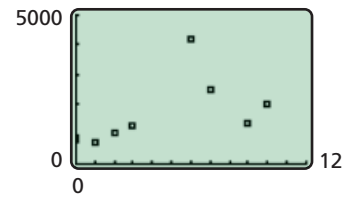
Step 1 Choose the degree of the polynomial model.

Let x represent the number of years since 1994. Make a scatter plot of the data.

The function appears to be cubic or quartic. Use the regression feature to check the R^2 -values.

cubic: $R^2 \approx 0.6279$ quartic: $R^2 \approx 0.8432$

The quartic function is a more appropriate choice.



Step 2 Write the polynomial model.

The data can be modeled by

$$f(x) = 9.27x^4 - 191.56x^3 + 1168.22x^2 - 1702.58x + 999.60$$

Step 3 Find the value of the model corresponding to 2002.

2002 is 8 years after 1994. Substitute 8 for x in the quartic model.

$$f(x) = 9.27(8)^4 - 191.56(8)^3 + 1168.22(8)^2 - 1702.58(8) + 999.60 \approx 2036.24$$

Based on the model, the opening value was about \$2036.24 in 2002.



3. Use a polynomial model to estimate the value of the index in 1999.

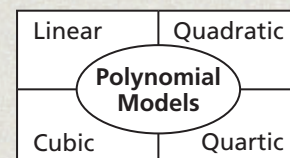
Year	1994	1995	1996	2000	2003	2004
Price (\$)	3754	3835	5117	11,497	8342	10,454

THINK AND DISCUSS

1. Suppose that finite differences are used to determine that a quartic polynomial best describes a particular data set. What is the minimum number of data pairs in the data set? Explain.



2. **GET ORGANIZED** Copy and complete the graphic organizer. For each type of function, indicate the degree and the constant differences and give an example of a data set.



GUIDED PRACTICE

SEE EXAMPLE 1

p. 466

Use finite differences to determine the degree of the polynomial that best describes the data.

1.

x	y
-2	22
-1	16
0	10
1	4
2	-2
3	-8

2.

x	y
-2	0
-1	-5
0	-3
1	5
2	18
3	35

3.

x	y
-2	23
-1	-2
0	1
1	2
2	-5
3	-2

SEE EXAMPLE 2

p. 467

4. **Business** The table below shows the number of square feet of retail space available for rent in various years. Write a polynomial function for the data.

Year	1957	1967	1977	1987	1997	2007
Retail Space (billion ft ²)	2.8	6.7	14.7	27.3	44.9	67.9

SEE EXAMPLE 3

p. 468

5. **Health** The table below shows the number of infected patients at various stages of a flu outbreak. Use a polynomial model to estimate the number of infected patients after 120 hours.

Time (h)	12	24	48	96	144	240
Patients	21	301	679	973	562	320

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
6-8	1
9	2
10-11	3

Extra Practice

Skills Practice p. S15

Application Practice p. S37

Use finite differences to determine the degree of the polynomial that best describes the data.

6.

x	y
-5	-20
-4	-19
-3	-9
-2	5.5
-1	20
0	30

7.

x	y
-2	-2
-1	-6
0	0
1	10
2	20
3	28

8.

x	y
-2	-3
-1	1
0	4.3
1	6.9
2	8.8
3	10

9. **Hobbies** The table below shows the number of Chess Club members in various years. Write a polynomial function for the data.

Year	2002	2003	2004	2005	2006	2007
Members	23	23	23	25	29	35



10. **Tourism** The table below shows the number of Canadian visitors to the United States. Use a polynomial model to predict the number of visitors in 2005.

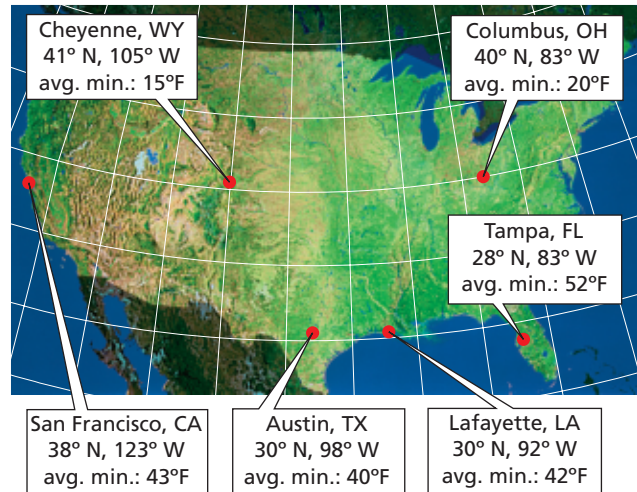
Year	1996	1998	2000	2001	2002	2003
Visitors (millions)	15.3	13.4	14.6	13.5	13.0	12.7

11. **Education** The table shows the total high school graduates in the United States. Use a polynomial model to estimate the total graduates in 1999.

Year	1989	1991	1995	1996	1998	2000
Graduates (thousands)	59,336	61,272	56,450	56,559	58,174	58,086

12. **Weather** The figure shows the latitude, longitude, and average January minimum temperature of various locations.

- Which pair of variables has a closer polynomial relationship: latitude and temperature or longitude and temperature? Give numerical data to support your answer.
- Is the relationship between latitude and temperature a function? Explain.
- Is a polynomial model relating longitude and temperature a function? Explain.



13. **Multi-Step** The table shows the total December clothing sales in the United States in billions of dollars.

Year	1997	1998	1999	2001	2002	2003
Sales (billions of \$)	12.1	12.7	13.5	14.1	14.6	15.3

- Write a cubic polynomial to model the data. What is the R^2 -value?
- Write a quartic polynomial to model the data. What is the R^2 -value?
- Is the difference in R^2 -values significant?
- What do the R^2 -values say about your answers to part **a** and part **b**?

**MULTI-STEP
TEST PREP**



14. This problem will prepare you for the Multi-Step Test Prep on page 472. You can make a pyramid by stacking balls in triangular layers. For example, three balls can be arranged as a triangle with a fourth ball on top of the other three.

Number of Balls on One Side of the Bottom Layer	1	2	3	4	5
Total Number of Balls in the Pyramid	1	4	10	20	35

- Use finite differences to determine the degree of the polynomial that best describes the data.
- Write a polynomial function for the data.
- How many balls are in a pyramid that has 12 balls on one side of the bottom layer?

15. **Critical Thinking** The fourth differences of a given data set are 0.01, 0, 0, and -0.01 . Is a cubic polynomial appropriate to model the data? Explain.
16. **Write About It** Describe the process you would use to determine the appropriate type of polynomial for a given data set.



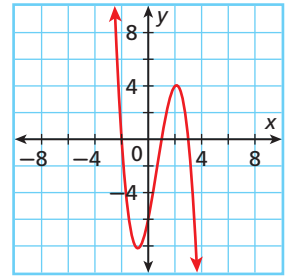
17. What type of polynomial best models the data below?

$$\{(0, 1), (1, 21), (2, 27), (3, 27), (4, 29), (5, 41)\}$$

- (A) Linear (B) Quadratic (C) Cubic (D) Quartic

18. Which cubic function represents the graph?

- (F) $f(x) = -(x + 2)(x - 1)(x - 3)$
 (G) $f(x) = (x + 2)(x - 1)(x - 3)$
 (H) $f(x) = -(x - 2)(x + 1)(x + 3)$
 (J) $f(x) = (x - 2)(x + 1)(x + 3)$



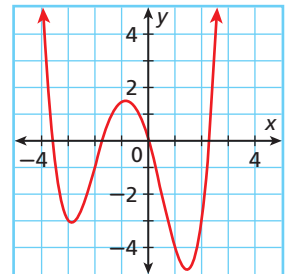
19. **Short Response** Give a cubic polynomial that can be used to model the data below.

$$\{(0, 4), (-1, 8), (1, 0), (-2, 6), (2, 2)\}$$

CHALLENGE AND EXTEND

The *average slope* of the graph of $f(x)$ between $x = a$ and $x = b$ is the slope of the straight line between $(a, f(a))$ and $(b, f(b))$. Use the graph at right to complete Exercises 20–22.

20. What is the average slope between $x = -2$ and $x = 2$?
21. How can you use first differences to find the average slope between each pair of points on the graph?
22. What happens to the average slope of the graph as it nears a maximum point?
23. Use finite differences to create a data set that could be modeled by a quartic polynomial.



SPIRAL REVIEW

Simplify each expression. (Lesson 1-4)

24. $-b^2(2b^2 + 5b - 3)$ 25. $a + 3a - 5a^2(6a - a)$ 26. $\frac{u^2 - v + 3v}{v(v^2 - u)}$

27. **Multi-Step** The chart below shows the number of copies sold (in thousands) of a new music CD for 5 weeks. Graph the data. Describe the parent function and the transformation that best approximates the data set. (Lesson 1-9)

Week	1	2	3	4	5
Copies Sold (thousands)	2	3	10	29	66

Identify the leading coefficient, degree, and end behavior. (Lesson 6-7)

28. $f(x) = x^4 - x^2 + 3x - 1$ 29. $f(x) = 4x^5 - x^3 + 10$ 30. $f(x) = -3x^6 + 5x$

MULTI-STEP TEST PREP



Applying Polynomial Functions

Pyramid Pile-Up You can build a pyramid by stacking blocks in layers. The blocks in each layer are arranged in a square, and the layers grow successively larger as shown below.

1. The figure shows the relationship between the number of layers and the total number of blocks in the pyramid. Make a table that shows the relationship for the first five layers.



1 layer
1 block



2 layers
5 blocks



3 layers
14 blocks

2. Use finite differences to determine the degree of the polynomial that best describes the data.
3. Write a polynomial function for the data.
4. The Great Pyramid in Giza, Egypt, has 201 layers. Use your function to estimate the number of blocks in the pyramid.
5. A pyramid contains a total of 285 blocks. How many layers are in the pyramid?
6. Is it possible to build a pyramid that uses exactly 811 blocks? Why or why not? Give an explanation in terms of the solutions to a polynomial equation.
7. A pyramid is known to contain at least 10,000 blocks. What is the minimum number of layers in the pyramid?



Quiz for Lessons 6-5 Through 6-9



6-5 Finding Real Roots of Polynomial Equations

- The yearly profit of a company in thousands of dollars can be modeled by $P(t) = t^4 - 10t^2 + 9$, where t is the number of years since 2000. Factor to find the years in which the profit was 0.

Identify the roots of each equation. State the multiplicity of each root.

- $x^3 + 6x^2 + 12x + 8 = 0$
- $2x^3 + 8x^2 - 32x - 128 = 0$
- $x^4 - 6x^3 + 9x^2 = 0$



6-6 Fundamental Theorem of Algebra

Write the simplest polynomial function with the given roots.

- 1, 1, 2
- $i, -1, 0$
- Solve $x^4 - 2x^3 + 6x^2 - 18x - 27 = 0$ by finding all roots.



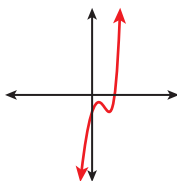
6-7 Investigating Graphs of Polynomial Functions

Graph each function.

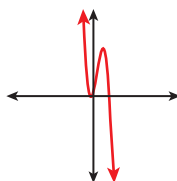
- $f(x) = x^4 - 13x^2 + 36$
- $f(x) = x^3 - 4x^2 - 15x + 18$

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.

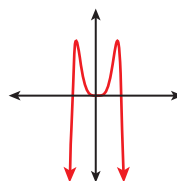
10.



11.



12.



6-8 Transforming Polynomial Functions

Let $f(x) = x^4 - 3x^2 + 6$. Write a function $g(x)$ that performs each transformation.

- Reflect $f(x)$ across the x -axis.
- Reflect $f(x)$ across the y -axis.

Let $f(x) = 8x^4 - 12x^2 + 2$. Graph $f(x)$ and $g(x)$ on the same coordinate plane.

Describe $g(x)$ as a transformation of $f(x)$.

- $g(x) = 3f(x)$
- $g(x) = f\left(\frac{1}{2}x\right)$
- $g(x) = f(x - 4)$



6-9 Curve Fitting with Polynomial Models

- The table shows the population of a bacteria colony over time. Write a polynomial function for the data.

Time (h)	1	2	3	4	5
Bacteria	44	112	252	515	949

Study Guide: Review

Vocabulary

degree of a monomial 406	local maximum 455	polynomial 406
degree of a polynomial 406	local minimum 455	polynomial function 408
end behavior 453	monomial 406	synthetic division 423
leading coefficient 406	multiplicity 439	turning point 455

Complete the sentences below with vocabulary words from the list above.

1. A(n) ? is a number or product of numbers and variables with whole number exponents.
2. A method of dividing a polynomial by a linear binomial of the form $x - a$ by using only the coefficients is ? .
3. The number of times $x - r$ is a factor of $P(x)$ is the ? of r .
4. The ? of a function is a description of the function values as x approaches positive infinity or negative infinity.

6-1 Polynomials (pp. 406–412)

EXAMPLES

- Subtract. Write your answer in standard form.

$$(6x - 2x^2 + 1) - (4x - 5x^2)$$

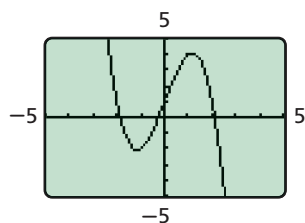
$$(-2x^2 + 6x + 1) + (5x^2 - 4x) \quad \text{Add the opposite.}$$

$$(-2x^2 + 5x^2) + (6x - 4x) + 1 \quad \text{Combine like terms.}$$

$$3x^2 + 2x + 1$$

- Graph $f(x) = -x^3 + 4x + 1$ on a calculator. Describe the graph, and identify the number of real zeros.

From left to right, the function decreases, increases, and then decreases again. It crosses the x -axis three times. There appear to be three real zeros.



EXERCISES

Rewrite each polynomial in standard form. Then identify the leading coefficient, degree, and number of terms. Name the polynomial.

5. $4x^2 - 3x^3 + 6x + 7$

6. $5x^3 - x^5 + 8x + 2x^4$

7. $1 - 11x + 9x^2$

8. $-6x^2 + x^4$

Add or subtract. Write your answer in standard form.

9. $(8x^3 - 4x^2 - 3x + 1) - (1 - 5x^2 + x)$

10. $(6x^2 + 7x - 2) + (1 - 5x^3 + 3x)$

11. $(5x - 2x^2) - (4x^2 + 6x - 9)$

12. $(x^4 - x^2 + 4) + (x^2 - x^3 - 5x^4 - 7)$

Graph each polynomial function on a calculator. Describe the graph, and identify the number of real zeros.

13. $f(x) = -x^4 + 4x^2 + 1$

14. $f(x) = x^3 + 2x^2 + 1$

15. $f(x) = x^4 - 5x^2 + 2$

16. $f(x) = x^3 - 3x^2 + 2$

6-2 Multiplying Polynomials (pp. 414–420)

EXAMPLE

- Find the product.

$$(x - 3)(5 - x - 2x^2)$$

Multiply horizontally.

$$(x - 3)(-2x^2 - x + 5) \quad \text{Write in standard form.}$$

$$x(-2x^2) + x(-x) + x(5) - 3(-2x^2) - 3(-x) - 3(5)$$

$$-2x^3 - x^2 + 5x + 6x^2 + 3x - 15 \quad \text{Multiply.}$$

$$-2x^3 + 5x^2 + 8x - 15 \quad \text{Combine like terms.}$$

EXERCISES

Find each product.

17. $5x^2(3x - 2)$

18. $-3t(2t^2 - 6t + 1)$

19. $ab^2(a^2 - a + ab)$

20. $(x - 2)(x^2 - 2x - 3)$

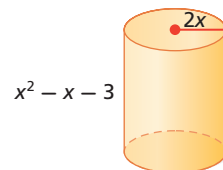
21. $(2x + 5)(x^3 - x^2 + 1)$

22. $(x - 3)^3$

23. $(x + 4)(x^4 - 3x^2 + x)$

24. $(2x + 1)^4$

25. A cylinder has a height of $x^2 - x - 3$ and a radius of $2x$ as shown. Express the volume of the cylinder as a sum of monomials.



6-3 Dividing Polynomials (pp. 422–428)

EXAMPLE

- Divide by using synthetic division.

$$(x^3 - 3x^2 + 8) \div (x + 2)$$

$$a = -2$$

$$x^3 - 3x^2 + 0x + 8 \quad \text{Write in standard form.}$$

$$\begin{array}{r|rrrr} -2 & 1 & -3 & 0 & 8 \\ & & -2 & 10 & -20 \\ \hline & 1 & -5 & 10 & -12 \end{array} \quad \text{Write the coefficients of the terms.}$$

$$\frac{x^3 - 3x^2 + 8}{x + 2} = x^2 - 5x + 10 + \frac{-12}{x + 2}$$

EXERCISES

Divide by using long division.

26. $(x^3 - 5x^2 + 2x - 7) \div (x + 2)$

27. $(8x^4 + 6x^2 - 2x + 4) \div (2x - 1)$

Divide by using synthetic division.

28. $(x^3 - 4x^2 + 3x + 2) \div (x - 3)$

29. $(x^3 + 2x - 1) \div (x - 2)$

30. A spool of ribbon has a length of $x^3 + x^2$ inches. Write an expression that represents the number of strips of ribbon with a length of $x - 1$ inches that can be cut from one spool.

6-4 Factoring Polynomials (pp. 430–435)

EXAMPLES

Determine whether each binomial is a factor of the polynomial $P(x) = 2x^2 + x - 10$.

- $(x + 5)$

$$\begin{array}{r|rrr} -5 & 2 & 1 & -10 \\ & & -10 & 45 \\ \hline & 2 & -9 & 35 \end{array}$$

$x + 5$ is not a factor of $P(x)$.

- $(x - 2)$

$$\begin{array}{r|rrr} 2 & 2 & 1 & -10 \\ & & 4 & 10 \\ \hline & 2 & 5 & 0 \end{array}$$

$x - 2$ is a factor of $P(x)$.

EXERCISES

Determine whether the given binomial is a factor of the polynomial $P(x)$.

31. $(x + 3)$; $P(x) = x^3 + 2x^2 - 5$

32. $(x - 1)$; $P(x) = 4x^4 - 5x^2 + 3x - 2$

33. $(x - 2)$; $P(x) = 2x^3 - 3x^2 + x - 6$

Factor each expression.

34. $x^3 - x^2 - 16x + 16$

35. $4x^3 - 8x^2 - x + 2$

36. $3x^3 + 81$

37. $16x^3 - 2$

6-5 Finding Real Roots of Polynomial Equations (pp. 438–444)

EXAMPLE

- Identify all of the real roots of $x^4 - 4x^3 + 4x^2 - 1 = 0$.

By the Rational Root Theorem, possible roots are ± 1 .

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 4 & 0 & -1 \\ & & 1 & -3 & 1 & 1 \\ \hline & 1 & -3 & 1 & 1 & 0 \end{array} \quad \text{Try 1.}$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 1 & 1 \\ & & 1 & -2 & -1 \\ \hline & 1 & -2 & -1 & 0 \end{array} \quad \text{Try 1 again.}$$

Factor $x^2 - 2x - 1$ by using the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = 1 \pm \sqrt{2}$$

The roots are 1 with a multiplicity of 2, and $1 \pm \sqrt{2}$.

EXERCISES

Identify all of the real roots of each equation.

38. $x^3 - 5x^2 + 8x - 4 = 0$
39. $x^3 + 6x^2 + 9x + 2 = 0$
40. $x^3 + 3x^2 + 3x + 1 = 0$
41. $x^4 - 12x^2 + 27 = 0$
42. $x^3 + x^2 - 2x - 2 = 0$
43. $x^3 - 5x^2 + 4 = 0$
44. A rectangular prism has length that is twice its width and height that is 4 meters longer than its width. The volume of the rectangular prism is 48 cubic meters. What is the width of the rectangular prism?

6-6 Fundamental Theorem of Algebra (pp. 445–451)

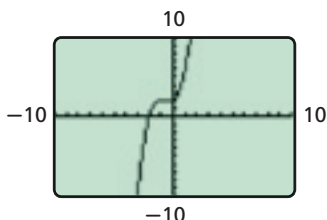
EXAMPLES

- Write the simplest polynomial function with roots -2 , -1 , and 4 .

$$\begin{aligned} P(x) &= 0 && \text{If } r \text{ is a root of } P(x), \\ a(x+2)(x+1)(x-4) &= 0 && \text{then } x-r \text{ is a} \\ &&& \text{factor of } P(x). \\ a(x^3 - x^2 - 10x - 8) &= 0 && \text{Multiply. For the} \\ x^3 - x^2 - 10x - 8 &= 0 && \text{simplest equation,} \\ &&& \text{let } a = 1. \end{aligned}$$

- Solve $x^3 + 2x^2 + x + 2 = 0$ by finding all roots.

The graphing calculator shows -2 as a root. Use synthetic division to write the equation as $(x+2)(x^2+1) = 0$. Solve $x^2+1=0$ to find the remaining roots. The solutions are -2 , i , and $-i$.



EXERCISES

Write the simplest polynomial function with the given roots.

45. $-3, 2, 4$
46. $-\frac{1}{2}, -2, 3$
47. $-\sqrt{2}, -1$
48. $-3, i$
49. $\sqrt{2}, \sqrt{3}$
50. $1 + \sqrt{3}, 2i$

Solve the equation by finding all roots.

51. $x^3 - x^2 + 4x - 4 = 0$
52. $x^4 - x^2 - 2 = 0$
53. $x^4 - \frac{63}{4}x^2 - 4 = 0$
54. $x^3 + 3x^2 - 5x - 15 = 0$

6-7 Investigating Graphs of Polynomial Functions (pp. 453–459)

EXAMPLE

- Graph the function $f(x) = x^3 + 2x^2 - 5x - 6$.

Leading coefficient: 1; Degree: 3;

End behavior: $x \rightarrow -\infty, f(x) \rightarrow -\infty$

$x \rightarrow +\infty, f(x) \rightarrow +\infty$

The zeros are $-3, -1, 2$. *Factor to find the zeros.*

$f(0) = -6; f(-2) = 4; f(1) = -8$ *Evaluate $f(x)$*

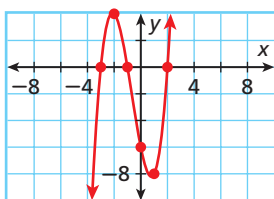
at values

between

the roots.

Plot these

points.



EXERCISES

Identify the leading coefficient, degree, and end behavior.

55. $-2x^3 + 5x^2 + 3$

56. $x^4 + 2x^3 - 3x + 1$

57. $-3x^6 + 9x^3 - 2x - 9$

58. $7x^5 + x^4 - 2x^2 + 5$

Graph each function.

59. $f(x) = x^3 - x^2 - 5x + 6$

60. $f(x) = x^4 - 10x^2 + 9$

61. $f(x) = -x^3 + 5x^2 + x - 5$

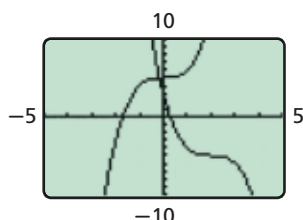
6-8 Transforming Polynomial Functions (pp. 460–465)

EXAMPLE

- Write a function that transforms $f(x) = x^3 + 5$ by reflecting it across the x -axis and shifting it 2 units right. Support your solution by using a graphing calculator.

$g(x) = -f(x - 2)$

$g(x) = -(x - 2)^3 - 5$



EXERCISES

Write a function that transforms $f(x) = x^4 - 6x^2 - 4$ in each of the following ways. Support your solution by using a graphing calculator.

62. Stretch vertically by a factor of 2, and move 9 units up.

63. Move 2 units down, and reflect across the x -axis.

64. Move 3 units right, and reflect across the y -axis.

6-9 Curve Fitting with Polynomial Models (pp. 466–471)

EXAMPLE

- The table shows the profit for a company in thousands of dollars for the years shown. Write a polynomial function for the data.

Year	1999	2000	2001	2002	2003
Profits	\$286	\$401	\$507	\$671	\$960

First differences: 115 106 164 289

Second differences: -9 58 125

Third differences: 67 67 *Constant*

A cubic polynomial best describes the data. Use the cubic regression feature on your graphing calculator.

$f(x) = 11.17x^3 - 38x^2 + 141.3x + 286$

EXERCISES

65. The chart shows the attendance for a new movie theater over five days. Write a polynomial function for the data.

Day	1	2	3	4	5
Attendance	248	298	318	388	428

66. The chart shows the population of a city for five years. Write a polynomial function for the data.

Year	1	2	3	4	5
Population (thousands)	1891	2674	3376	4480	6469

Add or subtract. Write your answer in standard form.

1. $(3x^2 - x + 1) + (x)$
2. $(6x^3 - 3x + 2) - (7x^3 + 3x + 7)$
3. $(y^2 + 3y^2 + 2) + (y^4 + y^3 - y^2 + 5)$
4. $(4x^4 + x^2) - (x^3 - x^2 - 1)$
5. The cost of producing x units of a product can be modeled by $C(x) = \frac{1}{10}x^3 - x^2 + 25$. Evaluate $C(x)$ for $x = 15$, and describe what the value represents.

Find each product.

6. $xy(2x^4y + x^2y^2 - 3xy^3)$
7. $(t + 3)(2t^2 - t + 3)$
8. $(x + 5)^3$
9. $(2y + 3)^4$

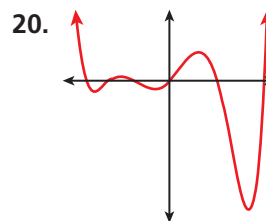
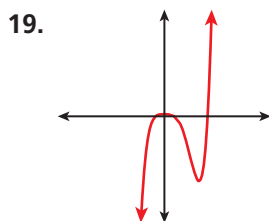
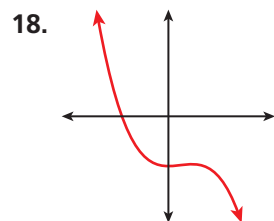
Divide.

10. $(5x^2 - 6x - 8) \div (x - 2)$
11. $(2x^3 - 7x^2 + 9x - 4) \div (2x - 1)$
12. Use synthetic substitution to evaluate $x^4 + 3x^3 - x^2 + 2x - 6$ for $x = 3$.

Factor each expression.

13. $-2x^2 - 6x + 56$
14. $m^5 + m^4 - 625m - 625$
15. $4x^3 - 32$
16. Identify the roots of the equation $2x^4 - 9x^3 + 7x^2 + 2x - 2 = 0$. State the multiplicity of each root.
17. Write the simplest polynomial function with roots of 1, 4, and -5 .

Identify whether the function graphed has an odd or even degree and a positive or negative leading coefficient.



Let $f(x) = 12x^3 + 4$. Graph $f(x)$ and $g(x)$ on the same coordinate plane. Describe $g(x)$ as a transformation of $f(x)$.

21. $g(x) = f(-x)$
22. $g(x) = \frac{1}{2}f(x)$
23. $g(x) = -f(x) + 3$
24. The table shows the number of bracelets Carly can make over time. Write a polynomial function for the data.

Time (h)	1	2	3	4	5	6
Bracelets	3	5	11	21	35	53

25. The table shows the number of sandwiches sold each day at a deli over 5 days. Write a polynomial function for the data.

Day	1	2	3	4	5
Sandwiches	57	72	101	89	66

FOCUS ON SAT MATHEMATICS SUBJECT TESTS

SAT Mathematics Subject Test results include scaled scores and percentiles. Your scaled score is a number from 200 to 800, calculated by using a formula that varies. The percentile indicates the percentage of people who took the same test and scored lower than you did.



The questions are written so that you should not need to do any lengthy calculations. If you find yourself getting involved in a long calculation, think again about all of the information in the problem to see if you might have missed something helpful.

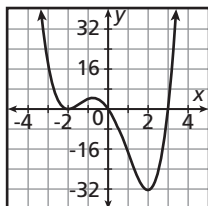
You may want to time yourself as you take this practice test. It should take you about 7 minutes to complete.

- If $x^4 - 7x^3 - 24x^2 + 112x + 128$ has a rational root a . Which could NOT be the value of a ?
(A) 0
(B) 16
(C) 24
(D) 32
(E) 64

- If there is a remainder of 3 when you divide $p(x) = x^3 + 4x^2 - hx + 30$ by $x - 3$, what is the value of h ?
(A) -90
(B) 4
(C) 12
(D) 27
(E) 30

- The graph of $q(x) = ax^4 + bx^3 + cx^2 + dx + f$ is shown below. Which of the following is true?

- $q(x)$ has an odd degree.
- $q(x) = (x + 3)h(x)$ for some polynomial $h(x)$.
- $f > 0$
- $q(0)$ is a local minimum.
- $q(x) \rightarrow \infty$ as $x \rightarrow \infty$



- Which of the following is the expanded form of $(3x + 2)^3$?
(A) $9x + 6$
(B) $27x^3 + 8$
(C) $27x^3 + 90x + 8$
(D) $27x^3 + 60x^2 + 30x + 8$
(E) $27x^3 + 54x^2 + 36x + 8$
- If $f(x)$ is a polynomial, which of the following transformations may affect the number of zeros of $f(x)$?
(A) Reflecting $f(x)$ across the y -axis
(B) Reflecting $f(x)$ across the x -axis
(C) Translating $f(x)$ 2 units to the right
(D) Translating $f(x)$ 6 units down
(E) Vertically stretching by a factor of 2

- Which of the following is a possible root of the polynomial $16x^4 + 80x^3 - 191x^2 + 8x + 15$?
(A) $\frac{1}{12}$
(B) $\frac{1}{5}$
(C) $\frac{3}{8}$
(D) $\frac{3}{5}$
(E) $\frac{8}{3}$



Any Question Type: Identify Key Words and Context Clues

When reading a test item, you should pay attention to key words and context clues in the problem statement. These clues will help you provide a correct response.

EXAMPLE 1

Short Response

Write a polynomial in standard form for the volume of the rectangular prism. Find the volume when $x = 5$ inches.

Look for key words and context clues. Identify what they mean.

Write a **polynomial** in **standard form** for the **volume** of the rectangular prism.

polynomial → a monomial or a sum or difference of monomials

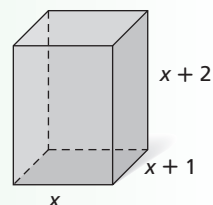
standard form → a polynomial written with its terms in descending order by degree

volume → volume of a rectangular prism ($V = \ell wh$) in cubic inches

$$V(x) = x(x + 1)(x + 2) = x^3 + 3x^2 + 2x \leftarrow \text{Standard form}$$

$$V(5) = 5^3 + 3(5^2) + 2(5) = 125 + 75 + 10 = 210 \text{ cubic inches}$$

The volume of the prism can be represented by $V(x) = x^3 + 3x^2 + 2x$, and when $x = 5$, $V = 210$ cubic inches.



EXAMPLE 2

Multiple Choice

Paige runs a small jewelry business. From 2000 through 2005, the number of items she created can be modeled by $24x + 12$, and the average cost to make each item can be modeled by $-0.05x^2 + 10$, where x is the number of years since 2000. Which polynomial can be used to model Paige's total jewelry-making costs for those years?

(A) $-0.05x^2 + 24x + 22$

(C) $-12x^3 - 6x^2 + 240x + 120$

(B) $0.05x^2 + 24x + 2$

(D) $-1.2x^3 - 0.6x^2 + 240x + 120$

The key words in this test item are **total cost**.

total cost → **average cost per unit** **times** the **number of units**

$$= -0.05x^2 + 10 \quad \times \quad 24x + 12$$

$$= -1.2x^3 - 0.6x^2 + 240x + 120$$

The correct answer is choice D.

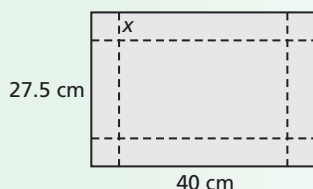


If you do not understand what a word means, reread the sentences around the word and make a logical guess.

Read each test item and answer the questions that follow.

Item A

Short Response A box can be made by cutting squares from each of the four corners of a piece of cardboard. The volume of a box made from a 27.5-by-40-centimeter piece of cardboard can be modeled by $x(27.5 - 2x)(40 - 2x)$, where x is the length of one side of the square. Write the volume as a sum of monomials, and find the volume when $x = 5$ centimeters.



1. What do $(27.5 - 2x)$ and $(40 - 2x)$ represent in the model?
2. Describe what "sum of monomials" means.
3. When you calculate the volume for $x = 5$, in what units should you give your response?

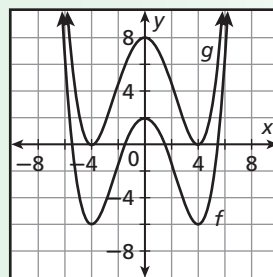
Item B

Short Response The volume of a cylindrical tank with a hemispherical top and bottom can be represented by the function $V(r) = 24\pi r^2 + \frac{4}{3}\pi r^3$, where V is the volume in cubic meters and r is the radius in meters. What is the radius if the volume of the tank is 5760π cubic meters?

4. Which word(s) in the problem statement tells you that the volume of a sphere is part of the function?
5. What does the term $24\pi r^2$ in the function represent?
6. Describe how to find the radius given the volume of the tank.

Item C

Multiple Choice Which description matches the transformation from f to g shown?



- (A) Vertical shift of 3 units
- (B) Vertical stretch by a factor of 3
- (C) Horizontal shift of 3 units
- (D) Horizontal stretch by a factor of 3

7. How do you know which is the original function and which is the image of the function?
8. Because the graphs are shown with an x - and y -scale of 1, how can you use the grid to identify a shift of 3 units?

Item D

Gridded Response A rectangular storage compartment has a length equal to its width and a height that is 5 feet greater than its width. The volume of the compartment is 72 cubic feet. What is the width?

9. Make a list of the key words given in the problem statement, and link each word to its mathematical meaning.
10. Write expressions representing the length and height of the compartment in terms of width.
11. Write an expression for the volume of the compartment.

CUMULATIVE ASSESSMENT, CHAPTERS 1–6

Multiple Choice

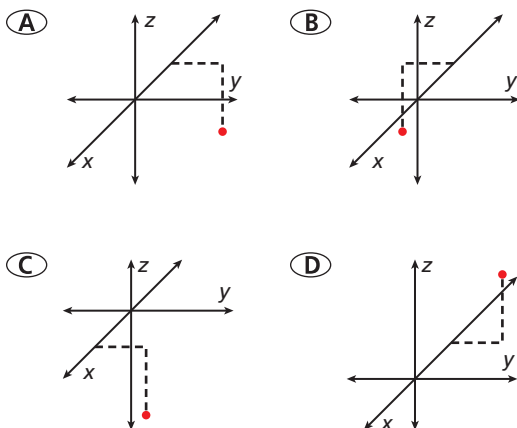
1. Which row of Pascal's triangle gives the coefficients for the binomial expansion of $(a + b)^4$?

(A) 1 2 1
 (B) 1 3 3 1
 (C) 1 4 6 4 1
 (D) 1 5 10 10 5 1

2. Which binomial is a factor of $2x^4 - 11x^3 + 19x^2 - 13x + 3$?

(F) $x - 2$ (H) $x + 1$
 (G) $x - 3$ (J) $x + 2$

3. Which graph shows the ordered triple $(-3, 3, -4)$ graphed in a three-dimensional coordinate plane?



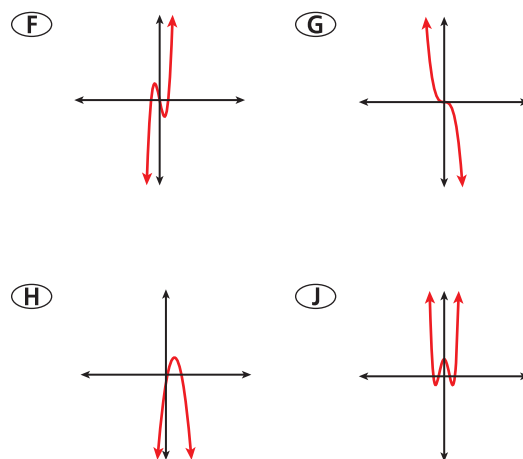
4. What is the value of the y-intercept of $2x + 4y = 1$?

(F) $\frac{1}{4}$ (H) 1
 (G) $-\frac{1}{2}$ (J) -2

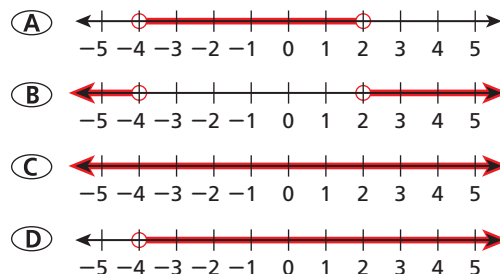
5. Simplify $\frac{4-i}{1+3i}$.

(A) $\frac{2}{5} - \frac{1}{10}i$ (B) $\frac{1}{10} - \frac{13}{10}i$
 (C) $\frac{7}{10} + \frac{11}{10}i$ (D) $4 - \frac{1}{3}i$

6. Which graph represents an odd degree polynomial function with a positive leading coefficient?



7. Solve the compound inequality $-3x < 12$ and $-8x + 1 < 9 - 12x$. Which of the following is a graph of the solution?



Use matrix N for Items 8 and 9.

$$N = \begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix}$$

8. What is the value of entry n_{21} ?

- (F) 2
(G) -3
(H) 4
(J) -6

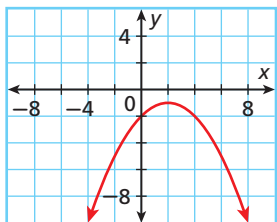
9. Find the inverse of matrix N .

- (A) $\begin{bmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ (C) $\begin{bmatrix} \frac{3}{2} & -2 \\ 2 & -1 \end{bmatrix}$
(B) $\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & -1 \\ \frac{3}{2} & -2 \end{bmatrix}$



In Item 10, use the end behavior of the graph to identify the sign of the leading coefficient and eliminate answer choices. Then use the y -intercept to choose the correct response.

10. What is the equation of the parabola shown?



- (F) $f(x) = 0.25x^2 + x - 2$
(G) $f(x) = 0.25x^2 - x - 2$
(H) $f(x) = -0.25x^2 + x + 2$
(J) $f(x) = -0.25x^2 + x - 2$

Gridded Response

11. What is the degree of the polynomial $7x^4 + 3x^2 - x^6 + 4$?

12. Divide by using long division. Identify the numerator of the remainder.

$$(2x^5 + 6x^4 - 10x^3 - 2x^2 + 54x + 14) \div (x - 4)$$

13. Complete the square for the expression $x^2 - 3x + \square$.

Short Response

14. The table below shows the number of spyware traces detected and removed from Larry's computer from January to June.

Jan	Feb	Mar	Apr	May	Jun
120	395	545	220	145	130

- a. Write a cubic function for the data.
b. Using your answer to part a, about how many traces can Larry expect to find in July?

15. Consider $5x^3 + 5x^2 - 40x - 60 = 0$.

- a. Identify the roots of the equation.
b. State the multiplicity of each root. Explain what the multiplicity means in terms of the graph.

16. Write the simplest polynomial function with zeros -1, 3, and 4.

17. Use synthetic substitution to evaluate $f(x) = x^3 + 5x^2 - 4x - 20$ for $x = -2$ and $x = 3$.

Extended Response

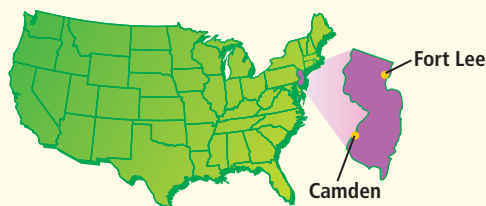
18. The functions g and h are the result of transformations of the function f .

- a. $f(x) = x^3 - 5x^2 + 8x - 1$ is reflected across the x -axis. Write the equation for $g(x)$.
b. Use a graphing calculator to graph f and g . Explain how the graph supports your answer to part a.
c. $f(x) = 4(x + 1)^3 + 4$ is translated 1 unit to the right and 3 units down. Write the equation for $h(x)$.
d. Use a graphing calculator to graph f and h . Explain how the graph supports your answer to part c.



Problem Solving on Location

NEW JERSEY



★ The Camden Waterfront

Located along the Delaware River, the Camden Waterfront offers a variety of attractions, including the Camden Riversharks minor league baseball team and the former home of poet Walt Whitman. The waterfront's most popular destination is the interactive Adventure Aquarium, which features two huge tanks: a 760,000-gallon open tank and a two-story, 550,000-gallon shark tank.

Choose one or more strategies to solve each problem.
For 1 and 2, use the table.

Adventure Aquarium		
Tank	Volume (gal)	Volume (m^3)
Open Ocean	760,000	2877
Shark Realm	550,000	2082

- The volume of a cylindrical tank can be modeled by the function $V(h) = \pi(h^3 + 6h^2 + 9h)$, where h is the tank's height in meters. Use this model to find the height of the Open Ocean tank.
- The Shark Realm tank has a viewing tunnel along its length. The volume of the tank can be modeled by $V(\ell) = \ell(\ell - 3.2)(\ell + 6.8)$, where ℓ is the tank's length in meters. What is the length of the viewing tunnel?
- The children's garden at the Camden Waterfront includes a 1200-square-foot room where visitors can walk among hundreds of flying butterflies. During one month of the year, the population of butterflies is modeled by $P(x) = 0.026x^3 - 1.3x^2 + 15.3x + 200$, where x is the number of days since the beginning of the month. What is the maximum population of butterflies during the month?





Problem Solving Strategies

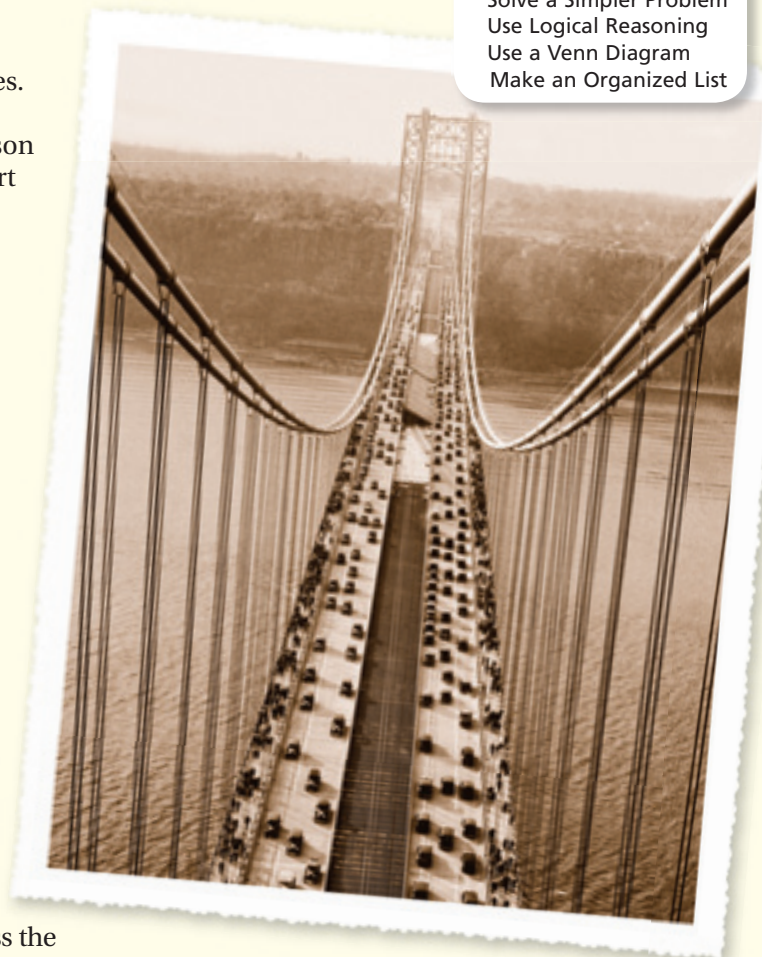
Draw a Diagram
Make a Model
Guess and Test
Work Backward
Find a Pattern
Make a Table
Solve a Simpler Problem
Use Logical Reasoning
Use a Venn Diagram
Make an Organized List

★ The George Washington Bridge

When the George Washington Bridge opened to traffic in 1931, it was the world's longest suspension bridge. Since then, it has been eclipsed by longer spans, but it remains one of the world's busiest bridges. On a typical day, more than 300,000 vehicles use the bridge to cross the Hudson River between Upper Manhattan and Fort Lee, New Jersey.

Choose one or more strategies to solve each problem.

1. The height in feet of the main cable above the roadway can be modeled by the equation $h = \frac{2}{15625}(x - 1750)^2$, where x is the distance in feet from the west tower. Find the height of the west tower.
2. The cable touches the roadway at the midpoint between the two towers. What is the length of the span between the towers?
3. The distance d traveled by a vehicle moving at an initial velocity of v_0 mi/h with a constant acceleration of a mi/h² is given by $d = \frac{1}{2}at^2 + v_0t$, where t is the time in hours. Given that the George Washington Bridge is 0.9 mi long, how long does it take a motorist to cross the bridge if she enters the bridge at 50 mi/h and accelerates at 10 mi/h²?



For 4, use the table.

4. The table shows the average hourly volume of eastbound traffic into Manhattan. Use a quadratic model to predict the hourly volume of traffic during the rush-hour peak from 6:00 A.M. to 7:00 A.M.

George Washington Bridge, Eastbound	
Hour	Average Vehicular Volume
3:00 A.M. to 4:00 A.M.	1274
4:00 A.M. to 5:00 A.M.	2035
5:00 A.M. to 6:00 A.M.	5581